A NUMERICAL METHOD FOR EQUATION OF MOTION IN DYNAMIC ANALYSIS OF DISCRETE STRUCTURES

by Nguyen Trong Phuoc[1] - Do Kien Quoc

ABSTRACT

In this paper, a time step integration method for resolving the differential equation of motion of discrete structures subjected to dynamic loads is presented. This method is derived based on the approximation of acceleration in two time steps by a combination of both trigonometric cosine and hyperbolic cosine functions with weighted coefficient. The necessary formula of the present method is elaborated for integrating of the governing equation of motion in structural dynamics. The accuracy and stability of the present method are also studied. The numerical results are compared with those obtained using Newmark method, linear acceleration method, showing high effectiveness of the new method.

Keywords: Numerical method, equation of motion, time step, acceleration, accuracy.

1. Introduction

For many structural problems, the evaluation of a structure using a static analysis may not be sufficient to obtain the actual response of the system; in this case dynamic analysis would be necessary [7, 13, 14]. Examples belong to diverse fields of structural dynamic problems such as infrastructures, buildings, offshore under dynamic loads derived from moving vehicles, landing impact upon aircraft, and natural causes such as wind, wave, and earthquake, etc. With mathematical models established from real structures, the governing equation can be obtained based on the balance of forces at time i for each degree of freedom [1]. To solve this problem, the governing equation of motion descretized by finite element methods becomes the second order ordinary differential equation. Due to the complexity of these equations, analytical solutions can only be obtained for a handful of simple problems [14]. Up to now, solutions to the equation of motion in the time domain are most conveniently obtained by computational techniques. Traditionally, time step integration methods are widely used in the framework of dynamic problems [2, 3, 5, 6] and others.

In the past several decades, several of time step integration methods have been introduced. In 1959, Newmark [10] introduced the family methods based on the variation of acceleration in each time step, which is well - known, in the field of dynamic analysis. Bathe and Wilson (1973) proposed the Wilson θ method and evaluated the accuracy, stability of solutions [1]. Hilber, Hughes, Taylor (1976) presented a method based on the equilibrium collocation, one parameter family of algorithms, higher order one step algorithms [7]. In 1980s, many authors suggested algorithms that can improve the effectiveness of computational process such as Hoff, Pahl (1988) with the implicit method with six free parameters θ_1, θ_2, θ_3, β, γ, η [8]. Recently, many authors developed the algorithms such as Hulbert, Mugan (2001) with the generalized α algorithm applied time domain, frequency domain and automatically chosen time step size; Fung (2003) with complex time step, higher order algorithms [5]; Xiaoqin Chen (1994) with Virtual Pulse method based

[1]*Department of Civil Engineering, Ho Chi Minh City University of Technology*

on a unique theoretical perspective with virtual displacement fields; Walker (2003, 2005) with higher order explicit - implicit algorithms by polynomial expressions to derive the final velocity and displacement equations [12, 13]; the higher - order accurate and unconditionally stable time-integration method by Kim et al. (1997) [9]; a nonlinear integration formula for ODEs proposed by Sivakumar et al. (1996) [12]; the combination of Newmark method and Wilson method applied in nonlinear problems used in ADINA software is introduced by Bathe (2005) [2]. From the above-listed methods, it can be seen that accurate and robust time step integration methods have been the focus of studies for fifty years, and are still under development.

With the assumption of variation of acceleration in two time steps by known nonlinear functions, implicit algorithms can be developed for equation of motion in structural dynamic problems. The objective of this paper is to deal with a new time step integration method for solving the equation of motion in structural dynamics. This method is derived based on the approximating acceleration by the combination of both trigonometric cosine and hyperbolic cosine functions with weighted coefficient in two time steps. The accuracy and stability of the proposed method are also studied. The numerical results for a single degree of freedom (DOF) system subjected to periodic loads with various frequencies are studied to verify the effectiveness of the new method.

2. Formulation

2.1. Equation of Motion

The governing equation of motion of a descretized structural model can be written as follows

$$(\mathbf{f}_I)_i + (\mathbf{f}_D)_i + (\mathbf{f}_S)_i = \mathbf{P}_i \qquad (1)$$

Here the vectors $(\mathbf{f}_I)_i$, $(\mathbf{f}_D)_i$, $(\mathbf{f}_S)_i$ and $\mathbf{P}_i$ are inertia force, damping force, spring or elastic force and external load vectors at time *i*, respectively. The external force is given by a set of discrete values $\mathbf{P}_i = \mathbf{P}(t_i)$, $i=0, 1, ..., n$. Time step $\Delta t = t_{i+1} - t_i$ is usually taken to be constant. The response is determined at the discrete time t_i, and denote u_i, $\dot{u}_i$ and $\ddot{u}_i$, respectively, the displacement, velocity and acceleration vectors as time *i*.

For linear dynamic problems, Eq. (1) can be represented as

$$M\ddot{u}_i + C\dot{u}_i + Ku_i = P_i \qquad (2)$$

where M, C and K are the mass, damping, and stiffness matrices, respectively. Consequently, the response of the system at time $i + 1$ can be described as follows

$$M\ddot{u}_{i+1} + C\dot{u}_{i+1} + Ku_{i+1} = P_{i+1} \qquad (3)$$

2.2. Time Step Integration Method

In this study, a new formulation of solving Eq. (2) is proposed by using a step by step integration technique. The acceleration function in two time steps is assumed by the combination of both trigonometric and hyperbolic cosine functions as shown in Figure 1 expressed as

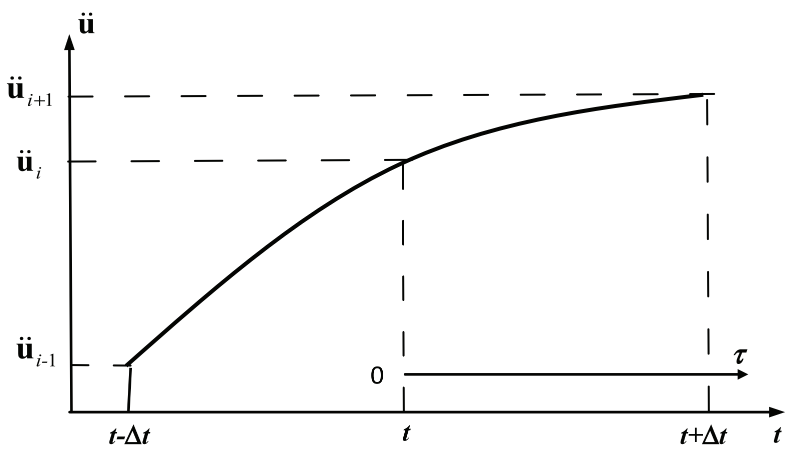
$$\begin{aligned}\ddot{\mathbf{u}}(t+\tau) = \theta\left(\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2} + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\tau - \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{2}\cos\frac{\pi\tau}{2\Delta t}\right) + \\ (1-\theta)\left(\ddot{\mathbf{u}}_i - \frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{\cosh(1)-1} + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\tau + \frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{\cosh(1)-1}\cosh\frac{\tau}{\Delta t}\right)\end{aligned} \qquad (4)$$

where Δt is the time step size, the time variable is $\tau \in [-\Delta t, \Delta t]$; the acceleration vectors at the times $t-\Delta t, t, t+\Delta t$ are defined as $\ddot{u}_{i-1}$, $\ddot{u}_i$, $\ddot{u}_{i+1}$, respectively; and Θ the weighted coefficient of trigonometric and hyperbolic cosine functions. It can be seen that Eq. (4) satisfies at time $\tau = -\Delta t$, $\tau = 0$, $\tau = \Delta t$ as follows

$$\ddot{\mathbf{u}}(t-\Delta t)=\theta\left(\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2}-\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\Delta t-\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{2}\cos\frac{-\pi\Delta t}{2\Delta t}\right)+$$
$$(1-\theta)\left(\ddot{\mathbf{u}}_{i}-\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}-\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\Delta t+\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\cosh\frac{-\Delta t}{\Delta t}\right)=\ddot{\mathbf{u}}_{i-1}$$
$$\ddot{\mathbf{u}}(t)=\theta\left(\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2}-\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{2}\right)+(1-\theta)\left(\ddot{\mathbf{u}}_{i}-\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}+\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\right)=\ddot{\mathbf{u}}_{i}$$
$$\ddot{\mathbf{u}}(t+\Delta\tau)=\theta\left(\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2}+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\Delta t-\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{2}\cos\frac{\pi\Delta t}{2\Delta t}\right)+$$
$$(1-\theta)\left(\ddot{\mathbf{u}}_{i}-\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{2\Delta t}\Delta t+\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\cosh\frac{\Delta t}{\Delta t}\right)=\ddot{\mathbf{u}}_{i+1}$$

(5)

Figure 1: The variation of acceleration in two time steps



Taking integration of Eq. (4), the resulting velocity equation can be expressed as

$$\dot{\mathbf{u}}(t+\tau)=\theta\left(\dot{\mathbf{u}}_{i}+\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2}\tau+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{4\Delta t}\tau^{2}-\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{2}\frac{2\Delta t}{\pi}\sin\frac{\pi\tau}{2\Delta t}\right)+$$
$$(1-\theta)\left(\dot{\mathbf{u}}_{i}+\ddot{\mathbf{u}}_{i}\tau-\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\tau+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{4\Delta t}\tau^{2}+\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\Delta t\sinh\frac{\tau}{\Delta t}\right)$$

(6)

Similarly, the displacement equation can be expressed by

$$\mathbf{u}(t+\tau)=\theta\left(\mathbf{u}_{i}+\dot{\mathbf{u}}_{i}\tau+\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{4}\tau^{2}+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{12\Delta t}\tau^{3}+\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{2}\frac{4\Delta t^{2}}{\pi^{2}}(\cos\frac{\pi\tau}{2\Delta t}-1)\right)+$$
$$(1-\theta)\left(\mathbf{u}_{i}+\dot{\mathbf{u}}_{i}\tau+\frac{1}{2}\ddot{\mathbf{u}}_{i}\tau^{2}-\frac{1}{4}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\tau^{2}+\frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{12\Delta t}\tau^{3}+\frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_{i}}{\cosh(1)-1}\Delta t^{2}(\cosh\frac{\tau}{\Delta t}-1)\right)$$

(7)

When $\tau=\Delta t$, the velocity and displacement vectors at the end time step are given as

$$\dot{\mathbf{u}}_{i+1} = \theta\left(\dot{\mathbf{u}}_i + \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{2}\Delta t + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{4}\Delta t - \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{2}\frac{2\Delta t}{\pi}\right) + (1-\theta)\left(\dot{\mathbf{u}}_i + \ddot{\mathbf{u}}_i\Delta t - \frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{\cosh(1)-1}\Delta t + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{4}\Delta t + \frac{1}{2}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{\cosh(1)-1}\Delta t\sinh(1)\right) \tag{8}$$

$$\mathbf{u}_{i+1} = \theta\left(\mathbf{u}_i + \dot{\mathbf{u}}_i\Delta t + \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}}{4}\Delta t^2 + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{12}\Delta t^2 - \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{2}\frac{4\Delta t^2}{\pi^2}\right) + (1-\theta)\left(\mathbf{u}_i + \dot{\mathbf{u}}_i\Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i\Delta t^2 - \frac{1}{4}\frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{\cosh(1)-1}\Delta t^2 + \frac{\ddot{\mathbf{u}}_{i+1}-\ddot{\mathbf{u}}_{i-1}}{12}\Delta t^2 + \frac{\ddot{\mathbf{u}}_{i+1}+\ddot{\mathbf{u}}_{i-1}-2\ddot{\mathbf{u}}_i}{2}\Delta t^2\right) \tag{9}$$

Substituting Eqs. (8), (9) into Eq. (3), the expression of the unknown $\ddot{u}_{i+1}$ can be obtained. Consequently, the velocity and displacement vectors at the end of time interval are determined by Eqs. (8), (9), respectively. The above-described process may be repeated to compute the dynamic response for subsequent discrete times.

2.3. Stability Analysis

The numerical stability of the numerical method is normally studied based on the mathematical theory. In this paper, the roots of the linear difference equation are applied to analyze the stability of the suggested method. Consider the linear difference equation as follows

$$a_n u_n + a_{n-1}u_{n-1} + ... + a_1 u_1 + a_0 u_0 = 0 \tag{10}$$

in which a_1, a_1, ..., a_n are constant coefficients. The auxiliary equation of Eq. (10), polynormial of variable λ, can be expressed as

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0 \tag{11}$$

The roots of the auxiliary Eq. (11), $\lambda_1, \lambda_2, ..., \lambda_n$ provide the values of which λ_1, $\lambda_2, ..., \lambda_n$ are needed to find u_i in accordance with Eq. (10).

The general solution of the linear difference equation can be determined as

$$u_i = C_1\lambda_1^i + C_2\lambda_2^i + ... + C_n\lambda_n^i$$

$$i = 0, 1, ..., \infty \tag{12}$$

in which C_1, C_2, ..., C_n are arbitrary constants to be determined from the specified initial conditions. Now investigating the stability of a method is based on the roots of the auxiliary equation $\lambda_1, \lambda_2, ..., \lambda_n$. Let $r(\ddot{e})$ be the spectral radius of the roots of the auxiliary λ_k, defined as

$$r(\ddot{e}) = \max\{r_k\}$$

$$k = 1, ..., n \tag{13}$$

where r_k are identical to the modulus of λ_k and determined as follows

$$r_k = |\lambda_k| \tag{14}$$

with $r(\ddot{e})$ are real or complex values

Time integration methods are unconditionally stable if the solution for any initial conditions does not grow without bound for any time step, in particular when time step is large. The method is conditionally stable if the same only holds provided time step is smaller than a certain value. It can be seen that u_i is bounded for $i\to\infty$ if and only if $r(\ddot{e})\leq 1$, and the solution is said stable, otherwise, the solution is unstable. Consequently, $r(\ddot{e})=1$ is considered as the stability limit criterion.

To study the stability properties of the formulae (8), (9), let us consider the free vibration response of a linear, undamped, single degree of freedom system governed by the following differential equation as follows

$$\ddot{u} + \omega^2 u = 0 \qquad (15)$$

where ω is the circular natural frequency. From the Eq.(9), application for the next time step gives as

$$u_{i+2} = \theta\left(u_{i+1} + \dot{u}_{i+1}\Delta t + \frac{\ddot{u}_{i+2} + \ddot{u}_i}{4}\Delta t^2 + \frac{\ddot{u}_{i+2} - \ddot{u}_i}{12}\Delta t^2 - \frac{\ddot{u}_{i+2} + \ddot{u}_i - 2\ddot{u}_{i+1}}{2}\frac{4\Delta t^2}{\pi^2}\right) +$$
$$(1-\theta)\left(u_{i+1} + \dot{u}_{i+1}\Delta t + \frac{1}{2}\ddot{u}_{i+1}\Delta t^2 - \frac{1}{4}\frac{\ddot{u}_{i+2} + \ddot{u}_i - 2\ddot{u}_{i+1}}{\cosh(1) - 1}\Delta t^2 + \frac{\ddot{u}_{i+2} - \ddot{u}_i}{12}\Delta t^2 + \frac{\ddot{u}_{i+2} + \ddot{u}_i - 2\ddot{u}_{i+1}}{2}\Delta t^2\right)$$

(16)

From the Eqs. (8), (9) and (16), the velocity vectors $\dot{u}_i$ and $\dot{u}_{i+1}$ are eleminated

$$u_{i+2} = 2u_{i+1} - u_i + \left\{\frac{\Delta t^2[(6\cosh(1) - 4\sinh(1) - 5)\ddot{u}_i + (2\sinh(1) - 2\cosh(1) + 1)\ddot{u}_{i-1}]}{4(\cosh(1) - 1)}\right\}(1-\theta) +$$
$$\left\{\frac{\Delta t^2[(2\cosh(1) - 3)\ddot{u}_{i+2} + (2\sinh(1) - 6\cosh(1) + 7)\ddot{u}_{i+1}]}{4(\cosh(1) - 1)} + \frac{\Delta t^2(\ddot{u}_{i+2} + 8\ddot{u}_{i+1} + 5\ddot{u}_i - 2\ddot{u}_{i-1})}{12}\right\}(1-\theta) +$$
$$\left\{\left(\frac{1}{3} - \frac{2}{\pi^2}\right)\ddot{u}_{i+2}\Delta t^2 + \left(\frac{5}{12} + \frac{6}{\pi^2} - \frac{1}{\pi}\right)\ddot{u}_{i+1}\Delta t^2 + \left(\frac{1}{6} - \frac{6}{\pi^2} + \frac{2}{\pi}\right)\ddot{u}_i\Delta t^2 + \left(\frac{1}{12} + \frac{2}{\pi^2} - \frac{1}{\pi}\right)\ddot{u}_{i-1}\Delta t^2\right\}\theta$$

(17)

By the substituting Eq. (15) into Eq.(17), the following difference equation is obtained as

$$\left\{\left[1 + \frac{\Omega^2}{3} - \frac{2\Omega^2}{\pi^2}\right]u_{i+2} - \left[2 - \frac{5\Omega^2}{12} - \frac{6\Omega^2}{\pi^2} + \frac{\Omega^2}{\pi}\right]u_{i+1} + \left[1 + \frac{\Omega^2}{6} - \frac{6\Omega^2}{\pi^2} + \frac{2\Omega^2}{\pi}\right]u_i + \left[\frac{\Omega^2}{12} + \frac{2\Omega^2}{\pi^2} - \frac{\Omega^2}{\pi}\right]u_{i-1}\right\}\theta +$$
$$\left\{\left[1 + \frac{\Omega^2}{12} + \frac{2\cosh(1) - 3}{4(\cosh(1) - 1)}\Omega^2\right]u_{i+2} - \left[2 - \frac{2\Omega^2}{3} - \frac{2\sinh(1) - 6\cosh(1) + 7}{4(\cosh(1) - 1)}\Omega^2\right]u_{i+1}\right\}(1-\theta) +$$
$$\left\{\left[1 + \frac{5\Omega^2}{12} + \frac{6\cosh(1) - 4\sinh(1) - 5}{4(\cosh(1) - 1)}\Omega^2\right]u_i + \left[-\frac{\Omega^2}{6} + \frac{2\sinh(1) - 2\cosh(1) + 1}{4(\cosh(1) - 1)}\Omega^2\right]u_{i-1}\right\}(1-\theta) = 0$$

(18)

in which $\Omega = \omega\Delta t$

Using cosh(1)=1.5430806; sinh(1)=1.1752012; π=3.141592654, Eq. (18) becomes

$$\left[(1 + 0.12299\Omega^2)u_{i+2} - (2 - 0.7089\Omega^2)u_{i+1} + (1 + 0.2131\Omega^2)u_i - 0.04503\Omega^2 u_{i-1}\right](1-\theta) +$$
$$\left[(1 + 0.1307\Omega^2)u_{i+2} - (2 - 0.7063\Omega^2)u_{i+1} + (1 + 0.1954\Omega^2)u_i - 0.03233\Omega^2 u_{i-1}\right]\theta = 0$$

(19)

This is homogeneous linear difference equation of third order. Consequently, the auxiliary equation of Eq. (19) can be written to be

$$\left[(1 + 0.1299\Omega^2)\lambda^3 - (2 - 0.7089\Omega^2)\lambda^2 + (1 + 0.2131\Omega^2)\lambda - 0.04503\Omega^2\right](1-\theta) +$$
$$\left[(1 + 0.1307\Omega^2)\lambda^3 - (2 - 0.7063\Omega^2)\lambda^2 + (1 + 0.1954\Omega^2)\lambda - 0.03233\Omega^2\right]\theta = 0$$

(20)

The roots of Eq. (20), λ_1, λ_2 and λ_3 are found and the spectral radius of the roots can be expressed in the table bellow

Table 1: The spectral radius with various time steps

Ω^2	$\approx \frac{T}{\Delta t}$	$\theta = 0$	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1$
0.05	28	1	1	1	1	1	1
0.10	20	1	1	1	1	1	1
0.20	14	1	1	1	1	1	1
0.3	11	1	1	1	1	1	1
0.4	10	1.00003	1.00003	1.00003	1.00003	1.00003	1.00003
∞	∞	∞	∞	∞	∞	∞	∞

Based on Table 1, it is seen that the new method is conditionally stable. The expression gives the condition for the stability as

$$\frac{\Delta t}{T} \leq \frac{1}{11} \qquad (21)$$

2.4. Accuracy Analysis

Based on the Taylor series expansion of the acceleration function at time *i,* the expansions of acceleration at the time *i*+1 and *i*-1 can be determined as follows

$$\ddot{\mathbf{u}}_{i+1} = \ddot{\mathbf{u}}_i + \dddot{\mathbf{u}}_i \Delta t + \frac{1}{2}\mathbf{u}_i^{IV}\Delta t^2 + \frac{1}{6}\mathbf{u}_i^{V}\Delta t^3 + 0(\Delta t^4)$$
$$\ddot{\mathbf{u}}_{i-1} = \ddot{\mathbf{u}}_i - \dddot{\mathbf{u}}_i \Delta t + \frac{1}{2}\mathbf{u}_i^{IV}\Delta t^2 - \frac{1}{6}\mathbf{u}_i^{V}\Delta t^3 + 0(\Delta t^4)$$

(22)

Substituting Eq. (22) into Eqs. (8), (9), the velocity and displacement at the end time interval of the suggested method may be expressed as

$$\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + \ddot{\mathbf{u}}_i \Delta t + \frac{1}{2}\dddot{\mathbf{u}}_i \Delta t^2 + \frac{1}{2}\left(\frac{\sinh(1)-1}{\cosh(1)-1}(1-\theta) + \frac{(\pi-2)}{\pi}\theta\right)\mathbf{u}_i^{IV}\Delta t^3 + 0(\Delta t^4)$$

(23)

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \dot{\mathbf{u}}_i \Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i \Delta t^2 + \frac{1}{6}\dddot{\mathbf{u}}_i \Delta t^3 + \left[\left(\frac{2\cosh(1)-3}{4(\cosh(1)-1)}\right)(1-\theta) + \left(\frac{\pi^2-8}{4\pi^2}\right)\theta\right]\mathbf{u}_i^{IV}\Delta t^4 + 0(\Delta t^5)$$

(24)

The Taylor series expansions of velocity and displacement at the end of time interval about at time *i* can be obtained as follows

$$\dot{\mathbf{u}}(i+1) = \dot{\mathbf{u}}_i + \ddot{\mathbf{u}}_i \Delta t + \frac{1}{2}\dddot{\mathbf{u}}_i \Delta t^2 + \frac{1}{6}\mathbf{u}_i^{IV}\Delta t^3 + 0(\Delta t^4)$$

(25)

$$\mathbf{u}(i+1) = \mathbf{u}_i + \dot{\mathbf{u}}_i \Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i \Delta t^2 + \frac{1}{6}\dddot{\mathbf{u}}_i \Delta t^3 + \frac{1}{24}\mathbf{u}_i^{IV}\Delta t^4 + 0(\Delta t^5)$$

(26)

Hence, the principal errors of velocity and displacement at the end time interval of the new method are given as follows

$$\mathbf{R}_{\dot{\mathbf{u}}} = \dot{\mathbf{u}}(i+1) - \dot{\mathbf{u}}_{i+1} = \left[\frac{1}{6} - \frac{1}{2}\left(\frac{\sinh(1)-1}{\cosh(1)-1}(1-\theta) + \frac{(\pi-2)}{\pi}\theta\right)\right]\mathbf{u}_i^{IV}\Delta t^3 + 0(\Delta t^4)$$

(27)

$$\mathbf{R}_{\mathbf{u}} = \mathbf{u}(i+1) - \mathbf{u}_{i+1} = \left[\frac{1}{24} - \left(\frac{2\cosh(1)-3}{4(\cosh(1)-1)}\right)(1-\theta) + \left(\frac{\pi^2-8}{4\pi^2}\right)\theta\right]\mathbf{u}_i^{IV}\Delta t^4 + 0(\Delta t^5)$$

(28)

For the comparison purpose, the truncation errors of velocity and displacement equations of Newmark method, linear acceleration method, are given as follows

$$\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + \ddot{\mathbf{u}}_i\Delta t + \frac{1}{2}\dddot{\mathbf{u}}_i\Delta t^2; \qquad \mathbf{T}_{\dot{\mathbf{u}}} = 0(\Delta t^3)$$

(29)

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \dot{\mathbf{u}}_i\Delta t + \frac{1}{2}\ddot{\mathbf{u}}_i\Delta t^2 + \frac{1}{6}\dddot{\mathbf{u}}_i\Delta t^3; \qquad \mathbf{T}_{\mathbf{u}} = 0(\Delta t^4)$$

(30)

It can be clearly seen that the proposed method is in good agreement with the Taylor series expansion up to the third order term of displacement or fourth order term based on the weighted coefficient θ.

In order to test the effectiveness of the presented formulation, a single degree of freedom systems is carried out in the next section. The comparison of the accuracy and convergence are given to illustrate the performance of the proposed method.

3. Numerical Example

The governing equation of motion of a single DOF system under periodic load is given as follows

$$\ddot{u}(t) + 2\zeta\,\omega\dot{u}(t) + \omega^2 u(t) = \frac{1}{m}p_0 \sin\omega_f\, t$$

(31)

with mass m = 1kg, natural frequency ω = 2πrad/s, damping ratio ζ, forcing amplitude and frequency p_0=5N and ω_f, ratio of frequencies $\beta = \frac{\omega_f}{\omega}$, initial conditions $u(0)=0, \dot{u}(0)=0$. The solutions of this problem are solved by analytical (exact) solution, Newmark solution (linear acceleration method), and suggested method.

For comparison goals, two parameters of the error are defined as follows:

The error of peak displacement

$$e_1 = \frac{A-\overline{A}}{\overline{A}}, \text{ and}$$

The average error per time step

$$e_2 = \frac{1}{N}\sum_{i=1}^{N}\left|u_i - \overline{u}_i\right|$$

Where $A, \overline{A}$ are the peak displacements of the calculated approximate solution and exact solution, $u_i, \bar{u}_i$ are the displacement of the calculated approximate solution and exact solution, and N is number of time steps.

Two cases with various damping ratio ζ and ratio of frequencies β are carried out as follows:

1. Given β =1.05; and ζ= 5%, the results including time history of displacement of exact solution, Newmark

solution, and suggested solution with time step $\Delta t = \frac{T}{10} = 0.0952$ s; error of peak displacement with various time steps from $\Delta t = \frac{T}{50}$ to $\Delta t = \frac{T}{10}$; and average error per time step with various weighted coefficients are shown in Figures 2, 3, 4.

2. The input data of the single degree of freedom system is given as β =1.05; and ζ = 5%. The results are presented in Figures 5, 6, 7.

Figure 2 shows the displacements of single degree of freedom system. Comparing to the exact solution, it can be seen that the present method gives very accurate solution. The result of convergence study is shown in Figure 4; the error of peak displacement derived from Newmark and suggested methods with various time steps are presented. This indicates that solutions obtained using the proposed method are more accurate than those obtained using Newmark method when the same time step is used. The best weighted coefficient is checked by numerical example; the survey of average error with the computational procedure about 18 periods in this example is expressed as Figure 3. It can be seen that the best weighted coefficient is the same as accuracy analysis section. The same comments are similar in the second example indicated in Figures 5, 6, 7.

Figure 2: Displacement of SDOF system with β =1.05; ζ= 5%; Δt = 0.0952 s = T/10

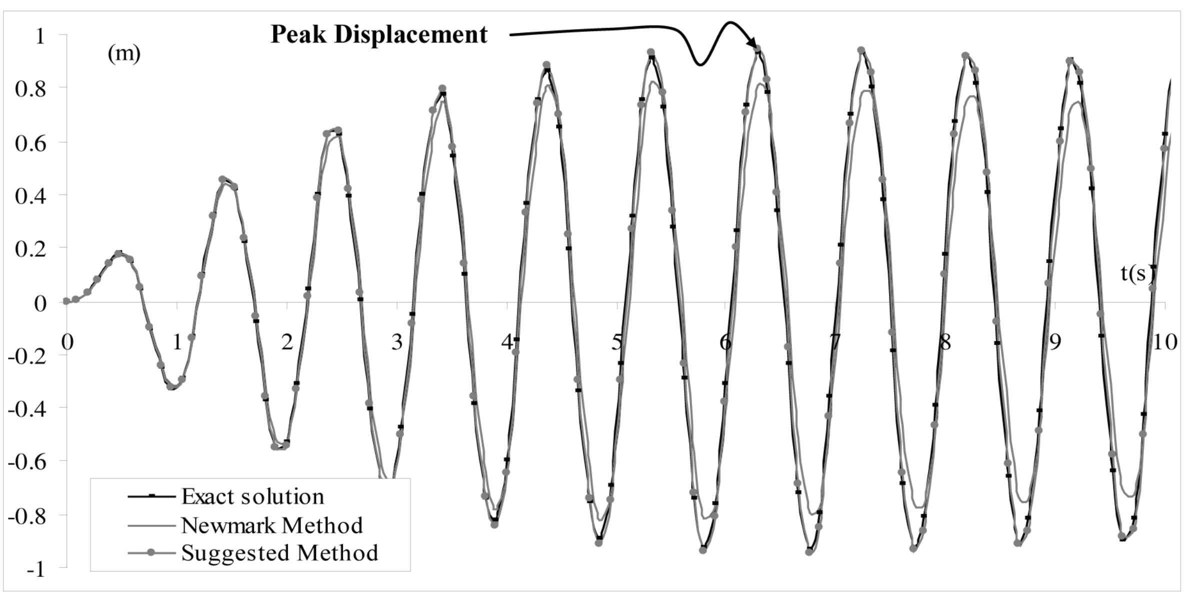


Figure 3: The average error per time step with Δt = 0.0474 s = T/20

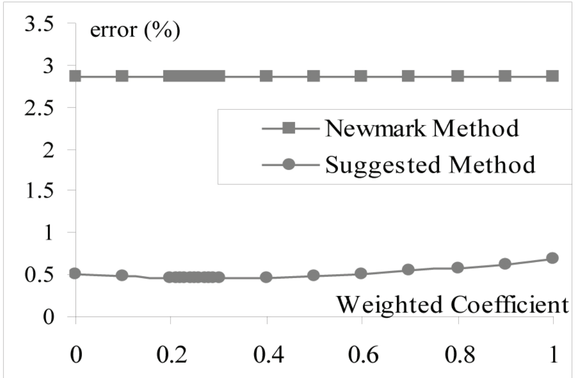


Figure 4: The error of peak displacement with various time steps

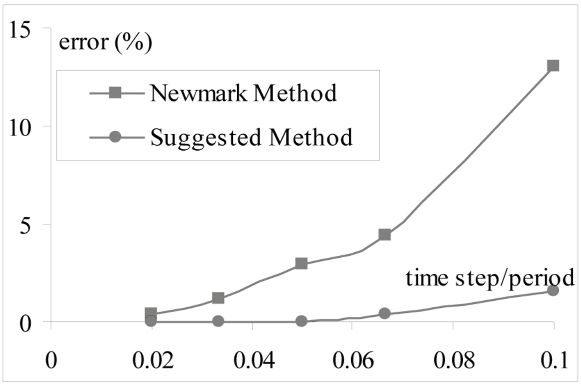


Figure 5: Displacement of SDOF system with β =1.5; ζ = 15%; Δt = 0.1s = T/6.666

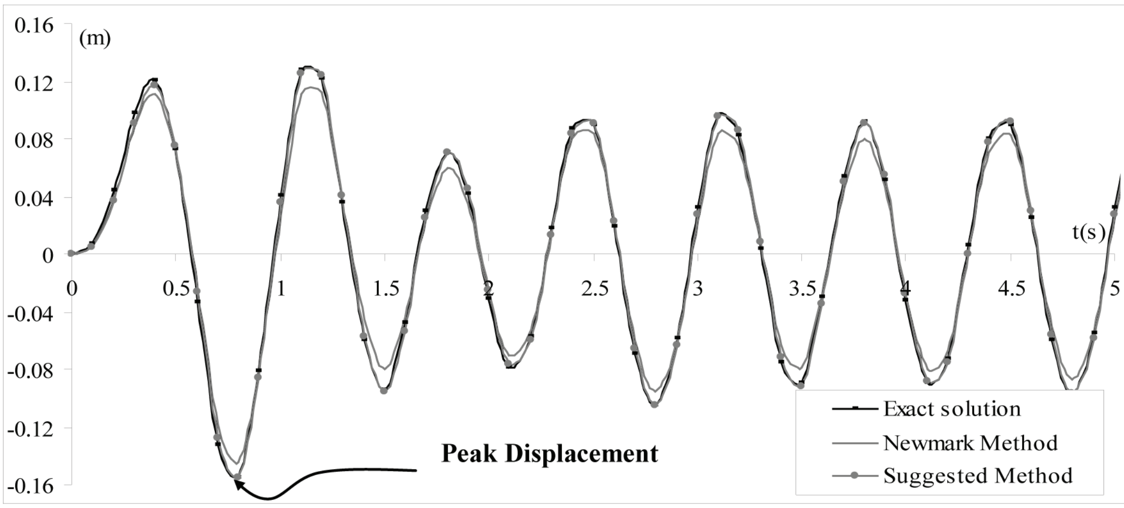


Figure 6: The average error per time step with Δt = 0.066 s = T/10

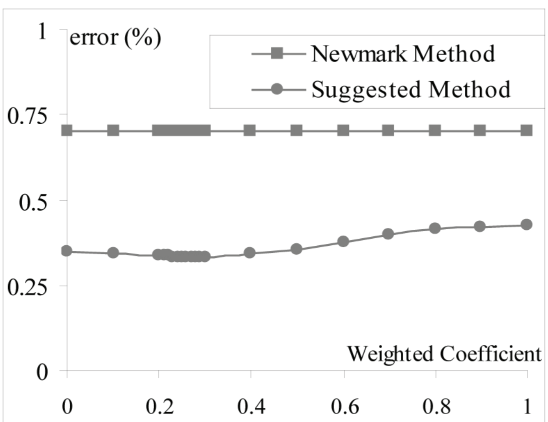
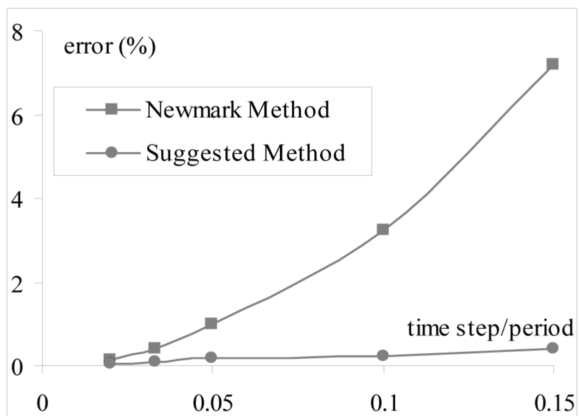


Figure 7: The error of peak displacement with various time steps



4. Conclusion

The numerical method of time step integration for the equation of motion in discrete structures under dynamic loads has been presented. The computational procedure of this method has been obtained from the approximation of acceleration in two time steps. The theoretical developments of this method included a detailed analysis of the stability, accuracy. The improved accuracy of this method based on the truncation errors from the Taylor series expansion was clearly evident from the theoretical developments. The numerical examples show that the computational performance of the present method is superior for dynamic problems.

REFERENCES

1. Bathe, K. J., and Wilson, E. L. (1973), "Stability and accuracy of direct integration methods", Earthquake Engineering and Structural Dynamics, 1, pp. 283-291.
2. Bathe, K. J., Irfan Baig, M. M. (2005), "On a composite implicit time integration procedure for nonlinear dynamics", Computers and Structures, 83, pp. 2513-2524
3. Do Kien Quoc, Nguyen Trong Phuoc (2005), "Solving dynamic equation using combination of both trigonometric and hyperbolic cosine functions for approximating acceleration", Journal of the Mechanical Science and Technology, 19 (special edi.), pp. 481-486.

4. Do Kien Quoc and Nguyen Trong Phuoc (2006), "A time step algorithm for dynamic analysis of structures due to earthquake", Proc. of the 10th East Asia-Pacific Conf. on Structural Engineering and Construction - EASEC10, 3. Wind and Earthquake Engineering, Thailand, pp. 429-434.
5. Fung, T.C. (2003), "Numerical dissipation in time step integration algorithms for structural dynamic analysis", Prog. Struct. Engng. Mater., 5, pp. 167-180.
6. Hahn G. D. (1991), "A modified Euler method for dynamic analyses", International Journal for Numerical Methods in Engineering, 32, pp. 932-955
7. Hilber, H. H. (1976), "Analysis and Design Time Integration Methods in Structural Dynamics", Report EERC 76-29, College of Engineering, University of California, Berkeley.
8. Hoff, C, and Pahl, P. J. (1988), "Development of an implicit method with numerical dissipation from a generalized single step algorithm for structural dynamics", Computer Methods in Applied Mechanics and Engineering, 67, pp. 67-385.
9. Kim S. J., Cho J. Y. and Kim W. D. (1997), "From the trapezoidal rule to higher-order accurate and unconditionally stable time-integration method for structural dynamics", Computer Methods in Applied Mechanics and Engineering, 149, pp. 73-88.
10. Newmark, N. M. (1959), "A method of computation for structural dynamics", Journal of Engineering Mechanic Division, Proceedings of the ASCE, pp. 63-95.
11. Sivakumar T. R., Savithri S. (1996), "A new nonlinear integration formula for ODEs", Journal of Computational and Applied Mechanics, 67, pp. 291-299.
12. Walke, Keierleber Colin (2003), Higher-order Explicit and Implicit Dynamic Time Integration Methods, Ph. D. Dissertation, University of Nebraska, USA.
13. Walker K. C., Rosson B. T. (2005), "Higher-order implicit dynamic time integration method", Journal of Structural Engineering, ASCE, 131, pp. 1267-1276.
14. Wang M., Au F. T. K. (2006), "Assessment and improvement of precise time step integration method", Computers and Structures, 84, pp. 779-786.