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NONLINEAR POST-BUCKLING OF THIN FGM ANNULAR SPHERICAL SHELLS UNDER MECHANICAL LOADS AND RESTING ON ELASTIC FOUNDATIONS

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Abstract. This paper presents an analytical approach to investigate the nonlinear buckling and post-buckling of thin annular spherical shells made of functionally graded materials (FGM) and subjected to mechanical load and resting on Winkler-Pasternak type elastic foundations. Material properties are graded in the thickness direction according to a simple power law distribution in terms of the volume fractions of constituents. Equilibrium and compatibility equations for annular spherical shells are derived by using the classical thin shell theory in terms of the shell deflection and the stress function. Approximate analytical solutions are assumed to satisfy simply supported boundary conditions and Galerkin method is applied to obtain closed-form of load-deflection paths. An analysis is carried out to show the effects of material and geometrical properties and combination of loads on the stability of the annular spherical shells.

Keywords: Nonlinear buckling, post-buckling, FGM annular spherical shells, mechanical loads, elastic foundations

1. INTRODUCTION

Considerable researches have focused on the thermo-elastic, dynamic and buckling analyses of functionally graded plates and shells in recent years. This is mainly due to the increased use of functionally graded materials (FGMs) as the components of structures in the advanced engineering. FGMs consisting of metal and ceramic constituents have received increasingly attention in structural applications. Smooth and continuous change in material properties enable FGMs to avoid interface problems and unexpected thermal stress concentrations. By high performance heat resistance capacity, FGMs are now chosen to use as structural components exposed to severe temperature conditions such as aircraft, aerospace structures, nuclear plants and other engineering applications.

On the other hand, the static and dynamic interaction of shells with elastic foundation is a problem of current importance. By using the theory of elasticity and theory of shells, have many approaches to analyze the interaction between a structure and an ambient medium. One of them is approach, which most earthen soils can be appropriately represented by a mathematical model from Pasternak, whereas sandy soils and liquids can be represented by Winkler's model ([1–3]).

As a result, the mechanical behavior of FGM shells, such as bending, vibration, stability, buckling, etc., have also been studied by many scientists. Notably among them are due to [4–10]. Recently, Duc et al. [11] investigated nonlinear axisymmetric response of FGM shallow spherical shells on elastic foundations, namely the Winkler-Pasternak foundations. Cheng and Batra [12] proposed a membrane analogy to derive an exact explicit eigenvalues for compression buckling, hydrothermal buckling, and vibration of FGM plates on a Winkler-Pasternak foundation based on the third-order plate theory. The free vibration, transient response, large deflection and post-buckling responses of FGM thin plates resting on Pasternak foundations were investigated by Yang and Shen [13] was using the method of differential quadrature and Galerkin procedure. Huang et al. [14] investigated Benchmark solutions for functionally graded thick plates resting onWinkler-Pasternak elastic foundations. Ying et al. [15] obtained two-dimensional elasticity solutions for functionally graded beams resting on elastic foundations. Sheng and Wang [16] analyzed thermal vibration, buckling and dynamic stability of functionally graded cylindrical shells embedded in an elastic medium. Sofiyev et al. investigated the vibration analysis of simply supported FGM truncated conical shells resting on the two-parameter elastic foundation [17], and [18] studied the buckling of the homogenous and FGM truncated conical shells resting on Winkler-Pasternak type elastic foundations. Shen [19] studied post-buckling of shear deformable FGM cylindrical shells surrounded by an elastic medium. Nonlinear free vibration response, static response under uniformly distributed load, and the maximum transient response under uniformly distributed step load of orthotropic thin spherical caps on elastic foundation have been obtained by Dumir [20].

The annular spherical shell is one of the special shapes of the spherical shells. To the best of our knowledge, there has been recently a few of publications on the annular shells. Alwar and Narasimhan [21] investigated the axisymmetric nonlinear analysis of laminated orthotropic annular spherical shell, the object of this investigation is to give analytical solutions of large axisymmetric deformation of laminated orthotropic spherical shells including asymmetric laminates. Dumir et al. [22] analyzed axisymmetric dynamic buckling analysis of laminated moderately thick shallow annular spherical cap under central ring load and uniformly distributed transverse load, applied statically or dynamically as a step function load. Eslami et al. [23] studied an exact solution for thermal buckling of annular FGM plates on an elastic medium and generalized coupled thermo-elasticity of functionally graded annular disk considering the Lord-Shulman theory [24].

In this study, by using the classical thin shell theory, an approximate solution, which was proposed by Agamirov [25] and was used by Sofiyev [18] for truncated conical shells, the authors tried to apply this form to solve problems related to annular FGM spherical shells. This study is one of the first attempts about the nonlinear postbuckling analysis of thin FGM annular spherical shells under mechanical loads and resting on Winkler-Pasternak type elastic foundations.

2. GOVERNING EQUATIONS

Consider an annular spherical shell is subjected to external pressure q uniformly distributed on the outer surface as shown in Fig. 1. R is the radius of curvature, r_1, r_0 indicate the radii and h is thickness of annular.

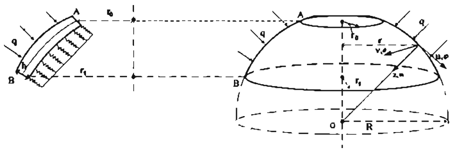

Fig. 1. Configuration of a FGM annular spherical shell

2.1. Material properties of functionally graded shells

The annular spherical shell is made of FGM, from a mixture of ceramics and metals, and is defined in coordinate system (φ, θ, z), where φ and θ are in the meridional and circumferential direction of the shells, respectively and z is perpendicular to the middle surface positive inwards.

Suppose that the material composition of the shell varies smoothly along the thickness by a simple power law in terms of the volume fractions of the constituents as

$$\begin{aligned} V_c(z) &= \left(\frac{2z+h}{2h}\right)^k, \quad -\frac{h}{2} \le z \le \frac{h}{2}, \\ V_m(z) &= 1 - V_c(z), \end{aligned} \tag{1}$$

where k (volume fraction index) is a non-negative number that defines the material distribution, subscripts m and c represent the metal and ceramic constituents, respectively.

The modulus of elasticity E of FGM annular spherical shell can be defined as

$$E(z) = E_m + E_{cm}\left(\frac{2z+h}{2h}\right)^k, \quad -\frac{h}{2} \le z \le \frac{h}{2}, \tag{2}$$

where the Poisson ratio ν is assumed to be constant $\nu(z) = const$ and $E_{cm} = E_c - E_m$.

In the present study, the classical shell theory is used to obtain the equilibrium and compatibility equations as well as expressions of buckling loads and nonlinear load-deflection curves of thin FGM annular spherical shells. For a thin annular spherical shell it is convenient to introduce a variable r, referred as the radius of parallel circle with the base of shell and defined by $r = R\sin\varphi$. Moreover, due to shallowness of the shell it is approximately assumed that $\cos\varphi = 1, Rd\varphi = dr$.

2.2. Winkler-Pasternak type elastic foundations

The annular spherical shell is resting on the elastic foundations. For the elastic foundation, one assumes the two-parameter elastic foundation model proposed by Pasternak [1]. The foundation medium is assumed to be linear, homogenous and isotropic. The bonding between the annular spherical shell and the foundation is perfect and frictionless. If the effects of damping and inertia force in the foundation are neglected, the foundation interface pressure may be expressed as

$$q_e = k_1 w - k_2 \Delta w,$$

where $\Delta w = \dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}$, w is the deflection of the annular spherical shell, k_1 is Winkler foundation modulus and k_2 is the shear layer foundation stiffness of Pasternak model.

2.3. Fundamental relations and governing equations

According to the classical shell theory, the strains at the middle surface and the change of curvatures and twist are related to the displacement components u, v, w in the φ, θ, z coordinate directions, respectively, taking into account Von Karman-Donnell nonlinear terms as [23]

$$\begin{aligned}
\varepsilon_r^0 &= \frac{\partial u}{\partial r} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2, & \chi_r &= \frac{\partial^2 w}{\partial r^2},\\
\varepsilon_\theta^0 &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} - \frac{w}{R} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2, & \chi_\theta &= \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2},\\
\gamma_{r\theta}^0 &= \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta}, & \chi_{r\theta} &= \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta},
\end{aligned} \tag{3}$$

where ε_r^0 and ε_θ^0 are the normal strains, γ_r^θ is the shear strain at the middle surface of the spherical shell, $\chi_r, \chi_\theta, \chi_{r\theta}$ are the changes of curvatures and twist.

The strains across the shell thickness at a distance z from the mid-plane are

$$\varepsilon_r = \varepsilon_r^0 - z\chi_r, \quad \varepsilon_\theta = \varepsilon_\theta^0 - z\chi_\theta, \quad \gamma_{r\theta}^0 = \gamma_r^\theta - z\chi_{r\theta}. \tag{4}$$

Using Eqs. (3) and (4), the geometrical compatibility equation of an shallow spherical shell is written as [10]

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_r^0}{\partial \theta^2} - \frac{1}{r}\frac{\partial \varepsilon_r^0}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varepsilon_\theta^0}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta}\left(r\gamma_{r\theta}^0\right) = -\frac{\Delta w}{r} + \chi_{r\theta}^2 - \chi_r\chi_\theta, \tag{5}$$

where $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$ is a Laplace's operator.

The stress-strain relationships are defined by the Hooke law

$$(\sigma_r, \sigma_\theta) = \frac{E(z)}{1-\nu^2}\left[(\varepsilon_r, \varepsilon_\theta) + \nu(\varepsilon_\theta, \varepsilon_r)\right], \sigma_{r\theta} = \frac{E(z)}{2(1+\nu)}\gamma_{r\theta}. \tag{6}$$

The force and moment resultants of an FGM spherical shell are expressed in terms of the stress components through the thickness as

$$(N_{ij}, M_{ij}) = \int_{-h/2}^{h/2} \sigma_{ij}(1, z)dz, ij = (rr, \theta\theta, r\theta) \tag{7}$$

In case of $(i = j = r)$ or $(i = j = \theta)$ for simplicity denoted $N_{rr} = N_r$, $N_{\theta\theta} = N_\theta$, $M_{rr} = M_r, M_{\theta\theta} = M_\theta$.

By using Eqs. (4), (6), and (7) the constitutive relations can be given as

$$\begin{aligned}
(N_r, M_r) &= \frac{(E_1, E_2)}{1-\nu^2}\left(\varepsilon_r^0 + \nu\varepsilon_\theta^0\right) - \frac{(E_2, E_3)}{1-\nu^2}\left(\chi_r + \nu\chi_\theta\right),\\
(N_\theta, M_\theta) &= \frac{(E_1, E_2)}{1-\nu^2}\left(\varepsilon_\theta^0 + \nu\varepsilon_r^0\right) - \frac{(E_2, E_3)}{1-\nu^2}\left(\chi_\theta + \nu\chi_r\right),\\
(N_{r\theta}, M_{r\theta}) &= \frac{(E_1, E_2)}{2(1+\nu)}\gamma_{r\theta}^0 - \frac{(E_2, E_3)}{1+\nu}\chi_{r\theta}.
\end{aligned} \tag{8}$$

From the relations one can write

$$\begin{aligned}
\varepsilon_r^0 = \frac{1}{E_1}(N_r - \nu N_\theta) + \frac{E_2}{E_1}\chi_r, \varepsilon_\theta^0 &= \frac{1}{E_1}(N_\theta - \nu N_r) + \frac{E_2}{E_1}\chi_\theta,\\
\gamma_{r\theta}^0 &= \frac{2(1+\nu)}{E_1}N_{r\theta} + \frac{2E_2}{E_1}\chi_{r\theta},
\end{aligned} \tag{9}$$

$$\begin{aligned}
M_r = \frac{E_2}{E_1}N_r - D(\chi_r + \nu\chi_\theta), M_\theta &= \frac{E_2}{E_1}N_\theta - D(\chi_\theta + \nu\chi_r),\\
M_{r\theta} &= \frac{E_2}{E_1}N_{r\theta} - D(1-\nu)\chi_{r\theta}.
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
D &= \frac{E_1E_3 - E_2^2}{E_1(1-\nu^2)},\ E_1 = \int_{-h/2}^{h/2} [E_c + E_{cm}(\frac{2z+h}{h})^k]dz = \left(E_m + \frac{E_{cm}}{k+1}\right)h,\\
E_2 &= \int_{-h/2}^{h/2} z[E_c + E_{cm}(\frac{2z+h}{h})^k]dz = h^2E_{cm}(\frac{1}{k+2} - \frac{1}{2k+2}),\\
E_3 &= \int_{-h/2}^{h/2} z^2[E_c + E_{cm}(\frac{2z+h}{h})^k]dz = \left(\frac{E_m}{12} + \frac{E_{cm}}{2(k+1)(k+2)(k+3)}\right)h^3.
\end{aligned} \tag{11}$$

The nonlinear equilibrium equations of a perfect shallow spherical shell based on the classical shell theory [11]

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r}{r} - \frac{N_\theta}{r} = 0, \tag{12}$$

$$\frac{\partial N_\theta}{r\partial \theta} + \frac{\partial N_{r\theta}}{\partial r} + \frac{2N_{r\theta}}{r} = 0, \tag{13}$$

$$\begin{aligned}&\frac{\partial^2 M_r}{\partial r^2} + \frac{2}{r}\frac{\partial M_r}{\partial r} + 2\Big(\frac{\partial^2 M_{r\theta}}{r\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial M_{r\theta}}{\partial\theta}\Big) + \frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial\theta^2} - \frac{1}{r}\frac{\partial M_\theta}{\partial r} + \frac{1}{R}(N_r + N_\theta) +\\ &+\frac{1}{r}\frac{\partial}{\partial r}\Big(rN_r\frac{\partial w}{\partial r} + N_{r\theta}\frac{\partial w}{\partial\theta}\Big) + \frac{1}{r}\frac{\partial}{\partial\theta}\Big(N_{r\theta}\frac{\partial w}{\partial r} + \frac{N_\theta}{r}\frac{\partial w}{\partial\theta}\Big) + q - k_1 w + k_2\Delta w = 0.\end{aligned} \tag{14}$$

The Eqs. (12), (13) are identically satisfied by introducing a stress function F as

$$N_r = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2},\ N_\theta = \frac{\partial^2 F}{\partial r^2},\ N_{r\theta} = -\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial F}{\partial\theta}. \tag{15}$$

Substituting Eqs. (3), (9), (15) into the Eqs. (5) and substituting Eqs. (3), (10), (15) into Eq. (14) leads to

$$\frac{1}{E_1}\Delta\Delta F = -\frac{\Delta w}{R} + \Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\Big)^2 - \frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big), \tag{16}$$

$$\begin{aligned}&D\Delta\Delta w - \frac{\Delta F}{R} - \Big(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial^2\theta}\Big)\frac{\partial^2 w}{\partial r^2} - \Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big)\frac{\partial^2 F}{\partial r^2}\\ &+2\Big(\frac{1}{r}\frac{\partial^2 F}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial F}{\partial\theta}\Big)\Big(\frac{1}{r}\frac{\partial^2 w}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial w}{\partial\theta}\Big) = q - k_1 w + k_2\Delta w.\end{aligned} \tag{17}$$

Regularly, the stress function F should be determined by the substitution of deflection function w into compatibility equation (16) and solving resulting equation. However, such a procedure is very complicated in mathematical treatment because obtained equation is a variable coefficient partial differential equation. Accordingly, integration to obtain exact stress function $F(r,\theta)$ is extremely complex. Similarly, the problem of solving the equilibrium is in the same situation. Therefore one should find a transformation to lead Eqs. (16), (17) into constant coefficient differential equations. Suppose such a transformation

$$w = w(\varsigma), F = F_0(\varsigma)e^{2\varsigma},\quad \text{where}\quad r = r_0 e^{\varsigma}, \varsigma = \ln\frac{r}{r_0}. \tag{18}$$

Substituting Eq. (18) into Eqs. (16), (17) and establishing a lot of calculations lead to the transformed equations

$$\begin{aligned}&\frac{1}{E_1}\left(\frac{\partial^4 F_0}{\partial\varsigma^4} + 4\frac{\partial^3 F_0}{\partial\varsigma^3} + 4\frac{\partial^2 F_0}{\partial\varsigma^2} + 4\frac{\partial^3 F_0}{\partial\varsigma\partial\theta^2} + 2\frac{\partial^4 F_0}{\partial\varsigma^2\partial\theta^2} + 4\frac{\partial^2 F_0}{\partial\theta^2} + \frac{\partial^4 F_0}{\partial\theta^4}\right)\\ &= -\frac{r_0^2}{R}\Big(\frac{\partial^2 w}{\partial\varsigma^2} + \frac{\partial^2 w}{\partial\theta^2}\Big) + \frac{1}{e^{4\varsigma}}\Big(\frac{\partial^2 w}{\partial\varsigma\partial\theta} - \frac{\partial w}{\partial\theta}\Big)^2 + \frac{1}{e^{4\varsigma}}\Big(\frac{\partial^2 w}{\partial\varsigma^2} - \frac{\partial w}{\partial\varsigma}\Big)\Big(\frac{\partial w}{\partial\varsigma} + \frac{\partial^2 w}{\partial\theta^2}\Big),\end{aligned} \tag{19}$$

$$
\begin{aligned}
& D\Big(\frac{\partial^4 w}{\partial \varsigma^4} - 4\frac{\partial^3 w}{\partial \varsigma^3} + 4\frac{\partial^2 w}{\partial \varsigma^2} - 4\frac{\partial^3 w}{\partial \varsigma \partial \theta^2} + 2\frac{\partial^4 w}{\partial \varsigma^2 \partial \theta^2} + 4\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4}\Big) \\
& - \frac{r_0^2 e^{4\varsigma}}{R}\Big(\frac{\partial^2 F_0}{\partial \varsigma^2} + 4\frac{\partial F_0}{\partial \varsigma} + 4F_0 + \frac{\partial^2 F_0}{\partial \theta^2}\Big) - \Big(\frac{\partial F_0}{\partial \varsigma} + 2F_0 + \frac{\partial^2 F_0}{\partial \theta^2}\Big)\Big(\frac{\partial^2 w}{\partial \varsigma^2} - \frac{\partial w}{\partial \varsigma}\Big)e^{2\varsigma} \\
& - \Big(\frac{\partial^2 F_0}{\partial \varsigma^2} + 2F_0 + 3\frac{\partial F_0}{\partial \varsigma}\Big)\Big(\frac{\partial w}{\partial \varsigma} + \frac{\partial^2 w}{\partial \varsigma \partial \theta}\Big)e^{2\varsigma} + 2\Big(\frac{\partial^2 F_0}{\partial \varsigma \partial \theta} + \frac{\partial F_0}{\partial \theta}\Big)\Big(\frac{\partial^2 w}{\partial \varsigma \partial \theta} - \frac{\partial w}{\partial \theta}\Big)e^{2\varsigma} \\
& - q r_0^4 e^{4\varsigma} + k_1 w r_0^4 e^{4\varsigma} - k_2\Big(\frac{\partial^2 w}{\partial \varsigma^2} + \frac{\partial^2 w}{\partial \theta^2}\Big) r_0^2 e^{2\varsigma} = 0.
\end{aligned} \tag{20}
$$

Eqs. (19) and (20) are the basic equations used to investigate the nonlinear buckling and postbuckling of FGM annular spherical shells resting on elastic foundations. These are nonlinear equations in terms of two dependent unknowns w and F.

3. STABILITY ANALYSIS

In this section, the FGM annular spherical shell is assumed to be simply supported along the periphery and subjected to mechanical loads uniformly distributed on the outer surface and the base edges of the shell. Depending on the in-plane behavior at the edge of boundary conditions will be considered in case the edges are simply supported and immovable. For this case, the boundary conditions are

$$
u = 0, w = 0, \frac{\partial^2 w}{\partial \varsigma^2} - \frac{\partial w}{\partial \varsigma} = 0, N_r = N_0, N_{r\theta} = 0, \quad \text{with } \varsigma = 0 \text{ (i.e at } r = r_0), \tag{21}
$$

where N_0 is the fictitious compressive load rendering the immovable edges.

The boundary conditions (21) can be satisfied when the deflection w is approximately assumed as follows [18, 25]

$$
w = W e^{\varsigma} \sin(\beta_1 \varsigma) \sin(n\theta), \quad \beta_1 = \frac{m\pi}{a}, \quad a = \ln \frac{r_1}{r_0}, \tag{22}
$$

where W is the maximum amplitude of deflection and m, n are the numbers of half waves in meridional and circumferential direction, respectively. The form of this approximate solution was proposed by Agamirov [25] and it was used by Sofiyev [18] for truncated conical shells.

Introduction of Eqs. (22) into Eq. (19) gives

$$
\begin{aligned}
& \frac{1}{E_1}\Big(\frac{\partial^4 F_0}{\partial \varsigma^4} + 4\frac{\partial^3 F_0}{\partial \varsigma^3} + 4\frac{\partial^2 F_0}{\partial \varsigma^2} + 4\frac{\partial^3 F_0}{\partial \varsigma \partial \theta^2} + 2\frac{\partial^4 F_0}{\partial \varsigma^2 \partial \theta^2} + 4\frac{\partial^2 F_0}{\partial \theta^2} + \frac{\partial^4 F_0}{\partial \theta^4}\Big) = \\
& = -\frac{r_0^2 e^{\varsigma} W}{R}\big[(1 - \beta_1^2 - n^2)\sin(\beta_1 \varsigma) + 2\beta_1 \cos(\beta_1 \varsigma)\big]\sin(n\theta) + \frac{W^2 \beta_1^2}{2}(n^2 - 1)\cos(2\beta_1 \varsigma) \\
& - \frac{W^2}{4}(\beta_1 - \beta_1 n - \beta_1^3)\sin(2\beta_1 \varsigma) + \frac{W^2}{2}\beta_1^2 n^2 \cos(2n\theta) \\
& + \frac{W^2}{4}(\beta_1 - \beta_1 n^2 - \beta_1^3)\sin(2\beta_1 \varsigma)\cos(2n\theta) + \frac{W^2}{2}\beta_1^2 \cos(2\beta_1 \varsigma)\cos(2n\theta).
\end{aligned} \tag{23}
$$

Solving this obtained equation with the boundary conditions (21) for the stress function F_0 yields

$$\begin{aligned} F_0 &= f_1 e^\varsigma \sin(\beta_1\varsigma)\sin(n\theta) + f_2 e^\varsigma \cos(\beta_1\varsigma)\sin(n\theta) + f_3\cos(2\beta_1\varsigma) + f_4\cos(2n\theta) \\ &+ f_5\cos(2\beta_1\varsigma)\cos(2\beta_2\theta) + f_6\sin(2\beta_1\varsigma)\cos(2n\theta) + f_7\sin(2\beta_1\varsigma) + {}^1\!/_2 N_0 r_0^2, \end{aligned} \tag{24}$$

with

$$\begin{aligned} f_1 &= -\frac{r_0^2 E_1 W[A(1-\beta_1^2-n^2)+2B\beta_1]}{(A^2+B^2)R}, f_2 = -\frac{r_0^2 E_1 W[B(1-\beta_1^2-n^2)+2A\beta_1]}{(A^2+B^2)R}, \\ f_3 &= E_1 l_3 W^2, f_4 = E_1 l_4 W^2, f_5 = E_1 l_4 W^2, f_6 = E_1 l_6 W^2, f_7 = E_1 l_7 W^2, \end{aligned} \tag{25}$$

and

$$\begin{aligned} A &= 9 - 22\beta_1^2 + \beta_1^4 + n^4 - 10n^2 + 2\beta_1^2 n^2, B = 8\beta_1^3 - 24\beta_1 + 8\beta_1 n^2, \\ l_3 &= \frac{b_6c_3 + c_4a_6}{a_3b_6 + a_6b_3}, l_4 = \frac{\beta_1^2}{32(\beta_2^2 - 1)}, l_5 = \frac{b_5c_5 + a_5c_6}{a_4b_5 + a_5b_4}, l_6 = \frac{a_4c_6 - b_4c_5}{a_4b_5 + a_5b_4}, l_7 = \frac{a_3c_4 - b_3c_3}{a_3b_6 + a_6b_3}. \end{aligned} \tag{26}$$

(the remaining constants are given in Appendix I)

Applying Galerkin method with the limits of integral is given by the formula

$$\int_0^{\ln\frac{r_1}{r_0}} \int_0^{2\pi} \Phi e^\varsigma \sin(\beta_1\varsigma)\sin(n\theta)d\varsigma d\theta = 0, \tag{27}$$

where Φ is the left hand side of Eq. (20) after substitution Eqs. (22) and (24) in it. From Eq. (27) we obtain the following equation

$$q - \frac{N_0A_5}{B_1r_0^2}W - \frac{A_0N_0}{RB_1} = \left(\frac{DA_3}{B_1r_0^4} + \frac{E_1A_4}{B_1R^2} + \frac{k_1A_6}{B_1} + \frac{k_2A_7}{B_1r_0^2}\right)W + \frac{E_1A_2}{RB_1r_0^2}W^2 + \frac{E_1A_1}{B_1r_0^4}W^3. \tag{28}$$

where the constants $B_1, A_0, A_1, A_2, A_3, A_4, A_5$ are given in Appendix II.

Eq. (28) is used to determine the buckling loads and nonlinear equilibrium paths of FGM annular spherical shell under mechanical loads including the effect of elastic foundations.

3.1. Case $N_0 = 0$

Eq. (28) reduces to

$$q = \left(\frac{D^*R_h^4A_3}{B_1R_0^4} + \frac{E_1^*A_4R_h^2}{B_1} + \frac{K_1D^*A_6}{B_1} + \frac{K_2D^*A_7R_h^2}{B_1R_0^2}\right)W^* + \frac{E_1^*A_2R_h^3}{B_1R_0^2}(W^*)^2 + \frac{E_1^*A_1R_h^4}{B_1R_0^4}(W^*)^3, \tag{29}$$

by putting

$$E_1^* = \frac{E_1}{h}, E_2^* = \frac{E_2}{h^2}, W^* = \frac{W}{h}, R_h = \frac{h}{R}, R_0 = \frac{r_0}{R}, D^* = \frac{D}{h^3}, K_1 = \frac{k_1h^4}{D}, K_2 = \frac{k_2h^2}{D}.$$

If the FGM annular spherical shell does not rest on elastic foundations, we received

$$q = \left(\frac{D^*R_h^4A_3}{B_1R_0^4} + \frac{E_1^*A_4R_h^2}{B_1}\right)W^* + \frac{E_1^*A_2R_h^3}{B_1R_0^2}(W^*)^2 + \frac{E_1^*A_1R_h^4}{B_1R_0^4}(W^*)^3. \tag{30}$$

Eqs. (29) and (30) may be used to find static critical buckling load and trace postbuckling load-deflection curves of FGM annular spherical shell. It is evident $q(W^*)$ curves originate from the coordinate origin. In addition, Eq. (29) indicates that there is no bifurcation-type buckling for pressure loaded annular spherical shell and extremum-type buckling only occurs under definite conditions. The extremum pressure buckling load of the shell can be found from Eqs. (29) and (30) using the condition $\dfrac{dq}{dW^*} = 0$.

3.2. Case $N_0 \neq 0$

A FGM annular spherical shell simply supported with immovable edges under simultaneous action of uniform external pressure q is considered.

The condition expressing the immovability on the boundary edges, i.e. $u = 0$ on $r = r_0$, and $r = r_1$ is fulfilled on the average sense as

$$\int_0^{\pi}\int_{r_0}^{r_1} \frac{\partial u}{\partial r} r dr d\theta = 0. \tag{31}$$

From Eqs. (3) and (9) one can obtain the following relation

$$\begin{aligned} \frac{\partial u}{\partial r} &= \varepsilon_r^0 + \frac{w}{R} - \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 \\ &= \frac{1}{E_1}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial r^2}\right) + \frac{E_2}{E_1}\frac{\partial^2 w}{\partial r^2} + \frac{w}{R} - \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2. \end{aligned} \tag{32}$$

Using the transformation (18) into Eq. (32) yields

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{E_1 r_0^2}\left[\frac{\partial F_0}{\partial \varsigma} + 2F_0 + \frac{\partial^2 F_0}{\partial \theta^2} - v\left(\frac{\partial^2 F_0}{\partial \varsigma^2} + 3\frac{\partial F_0}{\partial \varsigma} + 2F_0\right)\right] \\ &+ \frac{E_2}{E_1 r_0^2 e^{2\varsigma}}\left(\frac{\partial^2 w}{\partial \varsigma^2} - \frac{\partial w}{\partial \varsigma}\right) + \frac{w}{R} - \frac{1}{2r_0^2 e^{2\varsigma}}\left(\frac{\partial w}{\partial \varsigma}\right)^2 \end{aligned} \tag{33}$$

Introduction of Eqs. (22), (24) into the Eq. (33) then substituting obtained result into Eq. (31) lead to the expression for the fictitious load N_0

$$N_0 = \frac{E_1}{(-1+v)(-1+e^{2a})\pi r_0^2}\left[\frac{E_2 A_9}{E_1} + \frac{r_0^2 A_{10}}{R}\right] W + \frac{E_1 A_8}{(-1+v)(-1+e^{2a})\pi r_0^2} W^2, \tag{34}$$

where the constants A_8, A_9, A_{10} are given in Appendix II.

4. RESULTS AND DISCUSSION

In this section, the nonlinear response of the FGM annular spherical shell is analyzed. The shell is assumed to be simply supported along boundary edges. In characterizing the behavior of the spherical shell, deformations in which the central region of a shell moves toward the plane that contains the periphery of the shell are referred to as inward deflections (positive deflections). Deformations in the opposite direction are referred to as outward deflection (negative deflections).

The following properties of the FGM shell are chosen [11]: $E_m = 70\,\text{GPa}$, $E_c = 380\,\text{GPa}$, Poisson's ratio is chosen to be 0.3.

The effects of material and geometric parameters on the nonlinear response of the FGM annular spherical shells under mechanical loads including the effects of elastic foundation are presented in Figs. 2-6. It is noted that in all figures W/h denotes the dimensionless maximum deflection of the shell.

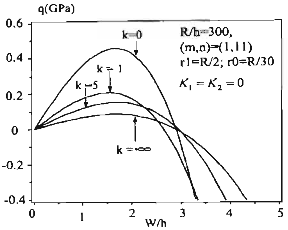

Fig. 2. Effects of volume fraction index k on the load-deflection curves of the FGM annular spherical shell under external pressure

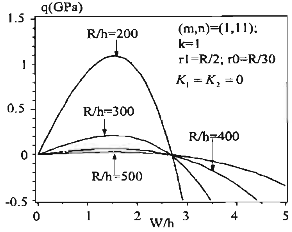

Fig. 3. Effects of curvature radius-thickness ratio on the nonlinear response of the FGM annular spherical shells under external pressure

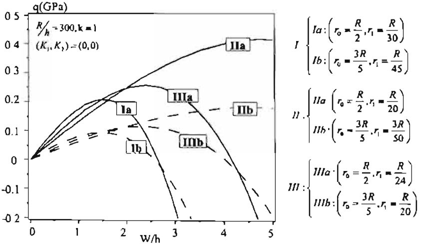

Fig. 4. Effects of radius of base-curvature radius ratio r_1/r_0 on the nonlinear response of the FGM annular spherical shells

Fig. 2 shows the effects of volume fraction index $k(0, 1, 5, +\infty)$ on the nonlinear response of the FGM annular spherical shell subjected to external pressure and compressive load (mode $(m, n) = (1, 11)$). As can be seen, the load-deflection curves become lower when

k increases. This is expected because the volume percentage of ceramic constituent, which has higher elasticity modulus, is dropped with increasing values of k.

Fig. 3 depicts the effects of curvature radius-thickness ratio R/h (200, 300, 400, and 500) on the nonlinear behavior of the external pressure and compressive load of the FGM annular spherical shells (mode $(m,n) = (1,11)$). From Fig. 3 we can conclude that when the annular spherical shells get thinner-corresponding with R/h getting bigger, the critical buckling loads will get smaller.

Fig. 4 analyzes the effects of 2 base-curvature radius ratio r_1/r_0 on the nonlinear response of FGM annular spherical shells subjected to uniform external pressure. It is shown that the nonlinear response of annular spherical shells is very sensitive with change of r_1/r_0 ratio characterizing the shallowness of annular spherical shell. Specifically, the enhancement of the upper buckling loads and the load carrying capacity in small range of deflection as r_1/r_0 increases is followed by a very severe snap-through behaviors. In other words, in spite of possessing higher limit buckling loads, deeper spherical shells exhibit a very unstable response from the post-buckling point of view. Furthermore, in the same effects of base-curvature radius ratio r_1/r_0 the load of the nonlinear response of FGM annular spherical shells is higher when the shallowness of annular spherical shell (H) is smaller, where H is the distance between two radii r_1, r_0, and calculated by

$$H(r_1,r_0) = \sqrt{R^2 - r_0^2} - \sqrt{R^2 - r_1^2} = R\left[\sqrt{1-\left(\frac{r_0}{R}\right)^2} - \sqrt{1-\left(\frac{r_1}{R}\right)^2}\right].$$

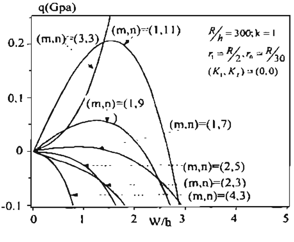

Fig. 5. Effects of mode (m,n) on the nonlinear response of the FGM annular spherical shells

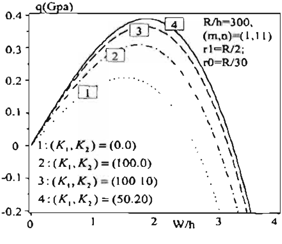

Fig. 6. Effects of the elastic foundations (K_1, K_2) on the nonlinear response of the FGM annular spherical shells

Fig. 5 examines the dependence of the nonlinear response of FGM annular spherical shells on the mode (m,n). It is easily recognized that with $m = 1$, the more increased the value of n, the higher increasing of the value of extreme point, corresponding to the higher load capacity of the shells. Note that, when m is even or $m \geq 3$, the graphic consists of symmetric curves through the origin of the coordinate system and the extreme point does not exist in the load-deflection curves.

Effects of the elastic foundations (K_1, K_2) on the nonlinear response of FGM annular spherical shells are shown in Fig. 6. Obviously, elastic foundations played positive role on nonlinear static response of the FGM annular spherical shell: the large K_1 and K_2 coefficients are, the larger loading capacity of the shells is. It is clear that the elastic foundations can enhance the mechanical loading capacity for the FGM annular spherical shells, and the effect of Pasternak foundation K_2 on critical uniform external pressure is larger than the Winkler foundation K_1.

5. CONCLUDING REMARKS

Due to practical importance of FGM annular spherical shells and the lack of investigations on stability of these structures, the present paper aims to propose an analytical approach to study the problem of nonlinear buckling and postbuckling of FGM thin annular spherical shells on elastic foundations. Based on the classical thin shell theory, the equilibrium and compatibility equations are derived in terms of the shell deflection and the stress function. This system of equations has been transformed into another system of more simple equations, so the appropriate formulas for FGM annular spherical shells are found as a special case. The results show the effects of the material composition, volume fraction of constituent materials, Winkler and Pasternak type elastic foundations on the nonlinear response of FGM annular spherical shells are very appreciable.

ACKNOWLEDGMENT

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APPENDIX I

$$a_3 = 16(\beta_1^4 - \beta_1^2), \qquad b_3 = 32\beta_1^3,$$
$$a_4 = 16(\beta_1^4 - \beta_1^2 + 2\beta_1^2 n - n^2 + n^4), \qquad b_4 = 32(\beta_1^3 + \beta_1 n^2),$$
$$a_5 = 32(\beta_1^3 + \beta_1 n^2), \qquad b_5 = 16(\beta_1 - \beta_1^2 + 2\beta_1^2 n^2 - n^2 + n^4),$$
$$a_6 = 32\beta_1^3, \qquad b_6 = 16(\beta_1^4 - \beta_1^2),$$
$$c_3 = 0.5(\beta_1^2 n^2 - \beta_1^2), \qquad c_5 = 0.5\beta_1^2,$$
$$c_4 = 0.25(\beta_1 n^2 - \beta_1 + \beta_1^3), \qquad c_6 = -c_4.$$

APPENDIX II

$$C_1 = a_4, \quad C_2 = b_5, \quad B_1 = \frac{-2m\pi a(-1 + e^{5a}(-1)^m)}{n(25a^2 + m^2\pi^2)}, \quad A_0 = \frac{m^2\pi^3\left(1 - e^{6a}\right)}{(9a^2 + m^2\pi^2)},$$

$$\begin{aligned}A_1 = {} & h_{11} + h_{12}\left(\frac{m\pi}{a}t_3 + t_4 - 2t_4 n^2\right) + h_{13}\left(\frac{m\pi}{a}t_6 + t_5\right) + h_{14}\left(\frac{m\pi}{a}t_5 + t_6\right) + \\ & + h_{15}\left(\frac{m\pi}{a}t_4 - t_3 - 2t_3 n^2\right) + h_{16}\left(\frac{2m^2\pi^2}{a^2}t_3 + \frac{3m\pi}{a}t_4 - t_3\right) + \\ & + h_{17}\left(\frac{-2m^2\pi^2}{a^2}t_4 + \frac{3m\pi}{a}t_3 + t_4\right) + h_{18}\left(\frac{2m^2\pi^2}{a^2}t_5 + \frac{3m\pi}{a}t_6 - t_5\right) + \\ & + h_{19}\left(\frac{-2m^2\pi^2}{a^2}t_6 + \frac{3m\pi}{a}t_5 + t_6\right) + h_{110}\left(t_3 - \frac{2m\pi}{a}t_4\right) + h_{111}\left(t_4 + \frac{2m\pi}{a}t_3\right),\end{aligned}$$

$$\begin{aligned}A_2 = {} & h_{21}\left(-6t_1 + \frac{m^2\pi^2}{a^2}t_1 + \frac{5m\pi}{a}t_2\right) + h_{22}\left(6t_2 - \frac{m^2\pi^2}{a^2}t_2 + \frac{5m\pi}{a}t_1\right) + \\ & + h_{23}\left(-t_3 + \frac{2m\pi}{a}t_4 + \frac{m^2\pi^2}{a^2}t_3 + n^2 t_3\right) + h_{24}\left(t_3 + \frac{2m\pi}{a}t_3 - \frac{m^2\pi^2}{a^2}t_4 + n^2 t_4\right) + \\ & + h_{25}\left(\frac{m^2\pi^2}{a^2}t_5 + \frac{2m\pi}{a}t_6 - t_5\right) + h_{26}\left(\frac{-m^2\pi^2}{a^2}t_6 + \frac{2m\pi}{a}t_5 + t_6\right) + h_{27} + \\ & + h_{28}\left(2nt_1 - \frac{mn\pi}{a}t_2\right) + h_{29}\left(2nt_2 + \frac{mn\pi}{a}t_1\right) + h_{210}\left(-3t_1 + \frac{m\pi}{a}t_2 + n^2 t_1\right) + \\ & + h_{211}\left(3t_2 + \frac{m\pi}{a}t_1 - n^2 t_2\right),\end{aligned}$$

$$A_3 = \frac{1}{8}\frac{m^2\pi^3\left(e^{2a} - 1\right)\left(a^4 + 2m^2\pi^2a^2 + m^4\pi^4 - 2n^2a^4 + 2m^2n^2\pi^2a^2 + n^4a^4\right)}{a^4(a^2 + m^2\pi^2)},$$

$$\begin{aligned}A_4 = {} & \frac{-m^2\pi^3\left(e^{6a} - 1\right)\left(-9t_1a^2 + 6t_2m\pi a + t_1m^2\pi^2 + t_1n^2a^2\right)}{24a^2(9a^2 + m^2\pi^2)} + \\ & + \frac{m\pi^2\left(e^{6a} - 1\right)\left(-9t_2a^2 - 6t_1m\pi a + t_2m^2\pi^2 + t_2n^2a^2\right)}{8a(9a^2 + m^2\pi^2)},\end{aligned}$$

$$A_5 = \frac{-8m^3\pi^3\left(-3m^2\pi^2 - 7a^2 + 3e^{5a}m^2\pi^2(-1)^m + 7e^{5a}(-1)^m a^2\right)}{3an(625a^4 + 250m^2\pi^2a^2 + 9m^4\pi^4)}.$$

$$A_6 = \frac{m^2\pi^3\left(e^{6a}-1\right)}{24\left(9a^2+m^2\pi^2\right)}, \quad A_7 = \frac{m^2\pi^3\left(-m^2\pi^2 + 3a^2e^{4a} - 3a^2 + e^{4a}m^2\pi^2\right)}{16a^2\left(4a^2+m^2\pi^2\right)},$$

$$A_8 = \frac{m^2\pi^2}{16a^2} - \frac{\left(t_6a^2 + t_5m\pi a - vt_6a^2 + 2vt_6m^2\pi^2 - 3vt_5m\pi a\right)}{a^2+m^2\pi^2} + \\ + \frac{m\pi\left(t_5a^2 - t_6m\pi a - vt_5a^2 + 2vt_5m^2\pi^2 + 3vt_6m\pi a\right)}{a\left(a^2+m^2\pi^2\right)}$$

$$A_9 = \frac{-\pi m\left(-1+e^a(-1)^m\right)}{an},$$

$$A_{10} = \frac{a\pi m\left(-1+e^{3a}(-1)^m\right)}{n\left(9a^2+m^2\pi^2\right)} \\ - \frac{m\pi\left(-1+e^{3a}(-1)^m\right)\left(-t_2m\pi a + 3t_1a^2 - n^2t_1a^2 + vm^2\pi^2t_1 + 5vt_2m\pi a - 6vt_1a^2\right)}{an\left(9a^2+m^2\pi^2\right)} \\ + \frac{3\left(-1+e^{3a}(-1)^m\right)\left(m\pi t_1a + 3t_2a^2 - n^2t_2a^2 - 5vm\pi t_1a + vt_2m^2\pi^2 - 6vt_2a^2\right)}{n\left(9a^2+m^2\pi^2\right)},$$

$$h_{11} = \frac{m^3\pi^4\left(e^{4a}-1\right)\left[4a^2\pi m\left(3n^2-1\right) + \left(2n^2-1\right)\left(m^3\pi^3-2a^3\right)\right]}{512a^4(n^2-1)(4a^2+m^2\pi^2)} + \\ + \frac{\pi^4m(e^{4a}-1)\left(a^3-\pi m^3\right)}{256a^2(n^2-1)(a^2+m^2\pi^2)},$$

$$h_{12} = \frac{-\pi^2m\left(e^{4a}-1\right)(2\pi m - a)}{32\left(a^2+m^2\pi^2\right)} + \frac{\pi^5m^4\left(e^{4a}-1\right)\left(\pi^2m^2-2a^2\right)}{32a^2(4a^4+5a^2m^2\pi^2+m^4\pi^4)},$$

$$h_{13} = \frac{\pi^3m^2\left(e^{4a}-1\right)(2\pi m - a)}{16a\left(a^2+m^2\pi^2\right)} + \frac{9m^7n\pi^{10}(-1+e^{4a})^2}{16a(4a^4+5a^2m^2\pi^2+m^4\pi^4)}$$

$$h_{14} = -2h_{12}, \quad h_{15} = -\frac{1}{2}h_{13}, \quad h_{16} = \frac{-3a\pi^4m^3(-1+e^{2a})}{8\left(a^4+5a^2m^2\pi^2+4m^4\pi^4\right)} + \frac{\pi^3m^2(-1+e^{2a})}{4\left(a^2+4m^2\pi^2\right)},$$

$$h_{17} = \frac{\pi^2m\left(e^{2a}-1\right)\left[2\pi^3m^3 - a^2\pi m + a^3 + am^2\pi^2\right]}{8\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)},$$

$$h_{18} = \frac{\pi^3m^2(-1+e^{2a})\left(3a\pi m - 2a^2 - 2\pi^2m^2\right)}{4\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)},$$

$$h_{19} = \frac{\pi^2m\left(e^{2a}-1\right)\left[-4\pi^3m^3 + 2a^2\pi m + a^3 + am^2\pi^2\right]}{8\left(a^2+4m^2\pi^2\right)\left(a^2+m^2\pi^2\right)}.$$

$$h_{110} = \frac{-\pi^4nm^3(-1+e^{4a})}{16a(a^2+m^2\pi^2)}, \quad h_{111} = \frac{n\pi^3m^2(-1+e^{4a})}{16(a^2+m^2\pi^2)},$$

$$h_{21} = \frac{8am^2\pi^2\left(a - m\pi\right)\left(-1 + (-1)^m e^{3a}\right)}{9n\left(9a^4 + 10a^2m^2\pi^2 + m^4\pi^4\right)},$$

$$h_{22} = \frac{4am\pi\left(-1 + (-1)^m e^{3a}\right)\left(-2am\pi + m^2\pi^2 + 3a^2\right)}{9n\left(9a^4 + 10a^2m^2\pi^2 + m^4\pi^4\right)},$$

$$h_{23} = \frac{160a^2m^2\pi^2(-1 + (-1)^m e^{5a})}{3n(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{24} = \frac{-8a\pi m\left(-1 + (-1)^m e^{5a}\right)\left(25a^2 - 3m^2\pi^2\right)}{3n(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{25} = \frac{160a^2m^2\pi^2}{3n(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{26} = -h_{24}, \quad h_{27} = \frac{am\pi^3\left(-1 + (-1)^m e^{5a}\right)}{12n(25a^2 + m^2\pi^2)},$$

$$h_{28} = \frac{-40\pi^3m^3a\left(-1 + (-1)^m e^{5a}\right)}{3(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{29} = \frac{4m^2\pi^2\left(-1 + (-1)^m e^{5a}\right)\left(3m^2\pi^2 + 25a^2\right)}{3(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{210} = \frac{-8m^2\pi^2\left(-1 + (-1)^m e^{5a}\right)\left(-5a^3 + 10a^2m\pi + 3m^3\pi^3\right)}{3an(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)},$$

$$h_{211} = \frac{4\pi m\left(-1 + (-1)^m e^{5a}\right)\left(50a^2m\pi - 25a^3 - 3am^2\pi^2 + 16m^3\pi^3\right)}{3n(625a^4 + 250a^2m^2\pi^2 + 9m^4\pi^4)};$$

$$t_1 = \frac{\left\{A(1 - \beta_1^2 - n^2) + 2B\beta_1\right\}}{(A^2 + B^2)},$$

$$t_2 = \frac{\left\{B(1 - \beta_1^2 - n^2) + 2A\beta_1\right\}}{(A^2 + B^2)},$$

$$t_3 = \frac{\left\{C_1(-\beta_1 n^2 + \beta_1 - \beta_1^3) - 64(\beta_1^3 + \beta_1 n^2)\beta_1^2\right\}}{4\left[C_1C_2 + (32\beta_1^3 + 32\beta_1 n^2)^2\right]},$$

$$t_4 = \frac{\left\{C_2\beta_1^2 + 16(\beta_1^3 + \beta_1 n^2)(-\beta_1 n^2 + \beta_1 - \beta_1^3)\right\}}{2\left[C_1C_2 + (32\beta_1^3 + 32\beta_1 n^2)^2\right]},$$

$$t_5 = \frac{4\left\{(\beta_1^4 - \beta_1^2)(\beta_1 n^2 - \beta_1 + \beta_1^3) - 4\beta_1^3(\beta_1^2 n^2 - \beta_1^2)\right\}}{(16\beta_1^4 - 16\beta_1^2)^2 + 1024\beta_1^6},$$

$$t_6 = \frac{8\left\{(\beta_1^4 - \beta_1^2)(\beta_1^2 n^2 - \beta_1^2) + (\beta_1 n^2 - \beta_1 + \beta_1^3)\beta_1^3\right\}}{(16\beta_1^4 - 16\beta_1^2)^2 + 1024\beta_1^6}.$$