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Some Results on Random Fixed Points of Completely Random Operators

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Abstract In this paper, some results on random fixed points of quasi-contractive and asymptotically contractive completely random operators are given. This is a continuation of the paper of Thang and Anh (Random Oper. Stoch. Equ. 21:1–20, 2013).

Keywords Random operator · Completely random operator · Lipschitz random operator · Random fixed point

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1 Introduction and Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X, Y be separable metric spaces and $F: \Omega \times X \to Y$ be a random operator in the sense that for each fixed x in X, the mapping $\omega \mapsto F(\omega, x)$ is measurable. An X-valued random variable ξ is said to be a random fixed point of the random operator $F: \Omega \times X \to X$ if $F(\omega, \xi(\omega)) = \xi(\omega)$ a.s. In recent years, many random fixed point theorems have been proved (see, e.g. [2–4] and the references therein). Some authors [3, 6, 8] have shown that under some assumptions, the random operator $F: \Omega \times X \to X$ has a random fixed point if and only if for almost all ω, the deterministic mapping $F_\omega: x \mapsto F(\omega, x)$ has a fixed point. Therefore, the existence of a random fixed point follows immediately from the existence of the corresponding deterministic fixed point.

A random operator $F: \Omega \times X \to Y$ may be considered as an action which transforms each deterministic input x in X into a random output $F(\omega, x)$ with values in Y. Taking into

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account many circumstances in which the inputs are also subject to influence of a random environment, an action which transforms each random input with values in X into random output with values in Y is called a completely random operator from X into Y.

As a continuation of [9], where some results about random fixed points of weakly contractive and semi-contractive completely random operators were presented, in this paper we obtain some results on random fixed points of quasi-contractive and asymptotically contractive completely random operators

2 Some Properties of Completely Random Operators

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and X be a separable Banach space. A mapping $\xi : \Omega \to X$ is called a X-valued random variable if ξ is $(\mathcal{F}, \mathcal{B})$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra of X. The set of all (equivalent classes) X-valued random variables is denoted by $L_0^X(\Omega)$ and it is equipped with the topology of convergence in probability Namely, the basis neighborhoods for this topology are the sets of the form $V(u_0, \epsilon, \alpha) = \{u \in L_0^X(\Omega) : \mathbb{P}\{\|u - u_0\| > \epsilon\} < \alpha\}$ and this topology is metrizable. The metric d on $L_0^X(\Omega)$ that induces this topology can be given by

$$d(u, v) = \mathbb{E}\frac{\|u - v\|}{1 + \|u - v\|}.$$

It is known that $L_0^X(\Omega)$ becomes a complete metric space under this metric (see [7]) and a sequence $(u_n) \subset L_0^X(\Omega)$ converges to u if and only if (u_n) converges to u in probability.

At first, recall that (see, e.g., [8]):

Definition 1 Let X, Y be two separable Banach spaces.

1. A mapping $F : \Omega \times X \to Y$ is said to be a random operator if for each fixed x in X, the mapping $\omega \mapsto F(\omega, x)$ is measurable.
2. A random operator $F : \Omega \times X \to Y$ is said to be continuous if for each ω in Ω the mapping $x \mapsto F(\omega, x)$ is continuous.

The following is the notion of a completely random operator.

Definition 2 (See [9]) Let X, Y be two separable Banach spaces.

1. A mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is called a completely random operator.
2. The completely random operator Φ is said to be continuous in probability if the mapping $\Phi : L_0^X(\Omega) \to L_0^Y(\Omega)$ is continuous, i.e., for each sequence (u_n) in $L_0^X(\Omega)$ such that $\lim_n u_n = u$ in probability, we have $\lim_n \Phi u_n = \Phi u$ in probability.
3. The completely random operator Φ is said to be an extension of a random operator F : $\Omega \times X \to Y$ if for each x in X

$$\Phi x(\omega) = F(\omega, x) \quad \text{a.s.},$$

where for each x in X, x denotes the random variable u in $L_0^X(\Omega)$ given by $u(\omega) = x$ a.s.

3 Random Fixed Points of Some Completely Random Operators

Let $F : \Omega \times X \to X$ be a random operator. Recall that (see e.g. [2–4]) an X-valued random variable ξ is said to be a random fixed point of the random operator F if

$$F(\cdot, \xi(\cdot)) = \xi(\cdot) \quad \text{a.s.}$$

Assume that F is continuous and let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be defined by $\Phi u(\omega) = F(\omega, u(\omega))$. Then by [9, Theorem 2.3] Φ is a completely random operator extending F and for each random fixed point ξ of F we have

$$\Phi\xi = \xi \quad \text{a.s.}$$

This leads to the following definition:

Definition 3 Let $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a completely random operator. An X-valued random variable ξ in $L_0^X(\Omega)$ is called a random fixed point of Φ if

$$\Phi\xi = \xi \quad \text{a.s.}$$

Next, we recall a notion of comparison function used by Beg in [1] and Olatinwo and Olaleru in [5]. This type of comparison function is used in order to extend the fixed point theorems satisfying contractive conditions.

Definition 4 [1, 5] A nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ is called a comparison function if

1. $f(t) = 0$ if and only if $t = 0$;
2. $\lim_{n\to\infty} f^n(t) = 0$ for all $t > 0$.

where $f^n(t) = \underbrace{f(f(\cdots f(t)\cdots))}_{n \text{ times}}$ and $f^0(t) = t$ for all $t \in [0, +\infty)$.

It is easy to see that the following lemma holds

Lemma 1 *If $f : [0, +\infty) \to [0, +\infty)$ is a comparison function then $f(t) < t$ for any $t > 0$.*

Definition 5 Given a comparison function $f : [0, +\infty) \to [0, +\infty)$ and a positive integer k. A continuous in probability completely random operator $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ is said to be (f, k)-quasi-contractive if

$$\mathbb{P}(\|\Phi^k u - \Phi^k v\| > t) \leq f(C(\Phi, u, v, t)) \tag{1}$$

for all u, v in $L_0^X(\Omega)$, $t > 0$ where

$$C(\Phi, u, v, t) = \max_{0\leq p, q\leq k, (p,q)\neq(k,k)} \{\mathbb{P}(\|\Phi^p u - \Phi^q v\| > t)\},$$

$\Phi^n u = \underbrace{\Phi(\Phi(\cdots\Phi(u)\cdots))}_{n \text{ times}}$ and $\Phi^0 u = u$ for all $u \in L_0^X(\Omega)$

Now, we are concerned with random fixed points of (f, k)-quasi-contractive completely random operators.

Theorem 1 *Let X be a separable Banach space and $\Phi : L_0^X(\Omega) \to L_0^X(\Omega)$ be a (f, k)-quasi-contractive completely random operator, where the comparison function f satisfies*

$$\sum_{i=1}^{\infty} f^i(1) < \infty. \tag{2}$$

Then, Φ has a unique random fixed point in $L_0^X(\Omega)$, and the iterative sequence $(\Phi^n u_0)$ converges in probability to a random fixed point of Φ for any random variable u_0 in $L_0^X(\Omega)$.

Proof Let u_0 be a random variable in $L_0^X(\Omega)$ and $u_{n+1} = \Phi u_n, n = 0, 1, \ldots$ From (1), for $n \geq k$ and all $t > 0$ we have

$$\begin{aligned}
\mathbb{P}(\|u_{n+1} - u_n\| > t) &= \mathbb{P}(\|\Phi^k(u_{n+1-k}) - \Phi^k(u_{n-k})\| > t) \\
&\leq f\Big(\max_{\substack{0 \leq p_1, q_1 \leq k, \\ (p_1, q_1) \neq (k,k)}} \{\mathbb{P}(\|\Phi^{p_1}(u_{n+1-k}) - \Phi^{q_1}(u_{n-k})\| > t)\}\Big) \\
&\leq \max_{\substack{0 \leq p_1, q_1 \leq k, \\ (p_1, q_1) \neq (k,k)}} \{f(\mathbb{P}(\|\Phi^{p_1}(u_{n+1-k}) - \Phi^{q_1}(u_{n-k})\| > t))\} \\
& \max_{\substack{0 \leq p_1, q_1 \leq k \\ (p_1, q_1) \neq (k,k)}} \{f(\mathbb{P}(\|u_{n+p_1+1-k} - u_{n+q_1+1-k}\| > t))\} \\
&\leq \cdots \\
&\leq \max_{\substack{0 \leq p_i, q_i \leq k, \\ (p_i, q_i) \neq (k,k)}} \{f^i(\mathbb{P}(\|u_{n+p_1+\cdots+p_i+1-ik} - u_{n+q_1+\cdots+q_i+1-ik}\| > t))\} \\
&\leq f^i(1)
\end{aligned}$$

with $i = [n/k]$. So, we have

$$\begin{aligned}
&\mathbb{P}(\|u_{n+h} - u_n\| > t) \\
&\quad \leq \mathbb{P}(\|u_{n+h} - u_{n+h-1}\| + \cdots + \|u_{n+1} - u_n\| > t) \\
&\quad \leq \mathbb{P}(\|u_{n+h} - u_{n+h-1}\| > t/h) + \cdots + \mathbb{P}(\|u_{n+1} - u_n\| > t/h) \\
&\quad < \sum_{i=[n/k]}^{[n+h/k]} f^i(1).
\end{aligned}$$

From (2), we have $\lim_n \sum_{i=[n/k]}^{[n+h/k]} f^i(1) = 0$ Hence, (u_n) is a Cauchy sequence in $L_0^X(\Omega)$. Then, there exists ξ in $L_0^X(\Omega)$ such that (u_n) converges in probability to ξ. Since $u_{n+1} = \Phi u_n$ and Φ is continuous in probability, letting $n \to \infty$ we get $\Phi\xi = \xi$, i.e., ξ is a random fixed point of Φ.

Let η be another random fixed point of Φ. So, for any $t > 0$, if $\mathbb{P}(\|\xi - \eta\| > t) > 0$, then from (1) we have

$$\begin{aligned}\mathbb{P}(\|\xi-\eta\|>t)&=\mathbb{P}(\|\Phi^k\xi-\Phi^k\eta\|>t)\\&\le \max_{0\le p,q\le k,(p,q)\ne(k,k)}\{f(\mathbb{P}(\|\Phi^p\xi-\Phi^q\eta\|>t))\}\\&=f(\mathbb{P}(\|\xi-\eta\|>t))\\&<\mathbb{P}(\|\xi-\eta\|>t)\end{aligned}$$

which yields a contradiction. So, we have $\mathbb{P}(\|\xi-\eta\|>t)=0$ for any $t>0$, i.e., $\xi=\eta$ a.s. Thus, Φ has a unique random fixed point. □

Taking $f(t)=\lambda t$, $0<\lambda<1$, we get

Corollary 1 *Let X be a separable Banach space, k be a positive integer number and Φ $L_0^X(\Omega)\to L_0^X(\Omega)$ be a continuous in probability completely random operator such that for each $t>0$, and u,v in $L_0^X(\Omega)$*

$$\mathbb{P}(\|\Phi^k u-\Phi^k v\|>t)\le\lambda C(\Phi,u,v,t),$$

where $0<\lambda<1$.

Then, Φ has a unique random fixed point in $L_0^X(\Omega)$, and the iterative sequence $(\Phi^n u_0)$ converges in probability to a random fixed point of Φ for any u_0 in $L_0^X(\Omega)$.

Example 1 Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, where $\Omega=[0,1]$, $\mathcal{F}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1]$, $\mathbb{P}$ is the Lebesgue measure on $[0,1]$ and $X=\mathbb{R}$.

Consider the completely random operator $\Phi: L_0^X(\Omega)\to L_0^X(\Omega)$ defined by

$$\Phi u(\omega)=\begin{cases}qu(2\omega) & \text{if } 0\le\omega\le\frac{1}{2},\\ 0 & \text{if } \frac{1}{2}<\omega\le 1,\end{cases}$$

where $q\in(0,1)$ is a real constant. Put

$$\begin{aligned}A&=\{\omega:\|\Phi u(\omega)-\Phi v(\omega)\|>t\}\\&=\left\{\omega\in\left[0,\frac{1}{2}\right]:\|u(2\omega)-v(2\omega)\|>t/q\right\}.\end{aligned}$$

and

$$B=\{\omega:\|u(\omega)-v(\omega)\|>t/q\}.$$

Then we see that B is the dilation of A, $B=2A$. So $\mathbb{P}(B)=2\mathbb{P}(A)$ and

$$\begin{aligned}\mathbb{P}(\|\Phi u(\omega)-\Phi v(\omega)\|>t)&=\frac{1}{2}\mathbb{P}(\|u(\omega)-v(\omega)\|>t/q)\\&\le\frac{1}{2}\mathbb{P}(\|u(\omega)-v(\omega)\|>t)\end{aligned}$$

for all $u,v\in L_0^X(\Omega)$ and $t>0$.

Hence,

$$\begin{aligned}\mathbb{P}(\|\Phi u(\omega)-\Phi v(\omega)\|>t)\\ \leq \frac{1}{2}\max\{\mathbb{P}(\|u(\omega)-v(\omega)\|>t), \mathbb{P}(\|u(\omega)-\Phi v(\omega)\|>t),\\ \mathbb{P}(\|\Phi u(\omega)-v(\omega)\|>t)\}\end{aligned}$$

and we find that Φ is (f,k)-quasi-contractive with $f(t)=t/2$ and $k=1$. It is easy to see that the random variable $u=0$ is a random fixed point of Φ.

Recall that if a subset A of $\mathbb{R}$ is Lebesgue measurable of measure λ and $\delta>0$, then the dilation of A by δ defined by $\delta A=\{\delta x : x\in A\}$ is also Lebesgue measurable of measure $\delta\lambda(A)$.

Definition 6 Let X be a separable Banach space, $f:[0,+\infty)\to[0,+\infty)$ be a comparison function. A continuous in probability completely random operator $\Phi: L_0^X(\Omega)\to L_0^X(\Omega)$ is said to be f-asymptotically contractive if there exist continuous functions $f_n:[0,+\infty)\to[0,+\infty)$ such that f_n uniformly converges to f and for all u,v in $L_0^X(\Omega), t>0$

$$\mathbb{P}(\|\Phi^n u-\Phi^n v\|>t)\leq f_n(\mathbb{P}(\|u-v\|>t)). \tag{3}$$

Theorem 2 *Let X be a separable Banach space and $\Phi: L_0^X(\Omega)\to L_0^X(\Omega)$ be a f-asymptotically contractive completely random operator. Then, Φ has a unique random fixed point and the iterative sequence $(\Phi^n u_0)$ converges in probability to a random fixed point of Φ for any u_0 in $L_0^X(\Omega)$*

Proof Let u,v be random variables in $L_0^X(\Omega)$. From (3), for $n\geq 1$ and $t>0$, we have

$$\mathbb{P}(\|\Phi^n u-\Phi^n v\|>t)\leq f_n(\mathbb{P}(\|u-v\|>t)).$$

So,

$$\begin{aligned}\limsup_n \mathbb{P}(\|\Phi^n u-\Phi^n v\|>t)&\leq \limsup_n f_n(\mathbb{P}(\|u-v\|>t))\\ &=f(\mathbb{P}(\|u-v\|>t)).\end{aligned}$$

Assume that $\limsup_n \mathbb{P}(\|\Phi^n u-\Phi^n v\|>t)=\epsilon>0$. From (3), we have

$$\mathbb{P}(\|\Phi^{n+k}u-\Phi^{n+k}v\|>t)\leq f_n(\mathbb{P}(\|\Phi^k u-\Phi^k v\|>t)).$$

It implies

$$\begin{aligned}&\limsup_n \mathbb{P}(\|\Phi^{n+k}u-\Phi^{n+k}v\|>t)\\ &\leq \limsup_n f_n(\mathbb{P}(\|\Phi^k u-\Phi^k v\|>t))\\ &=f(\mathbb{P}(\|\Phi^k u-\Phi^k v\|>t)).\end{aligned}$$

Then, we obtain

$$\limsup_{k}\limsup_{n}\mathbb{P}\big(\big\|\Phi^{n+k}u-\Phi^{n+k}v\big\|>t\big)$$
$$\leq \limsup_{k} f\big(\mathbb{P}\big(\big\|\Phi^{k}u-\Phi^{k}v\big\|>t\big)\big).$$

Hence, $0<\epsilon\leq f(\epsilon)<\epsilon$, a contradiction. So, assume that $\limsup_n \mathbb{P}(\|\Phi^n u-\Phi^n v\|>t)=0$.

Let u_0 be a random variable in $L_0^X(\Omega)$. Taking $u=\Phi^k u_0$, $v=u_0$, we deduce

$$\limsup_{n}\mathbb{P}\big(\big\|\Phi^{n+k}u_0-\Phi^{n}u_0\big\|>t\big)=0.$$

Putting $u_n=\Phi^n u_0$, it follows that (u_n) is a Cauchy sequence in $L_0^X(\Omega)$. Then, there exists ξ in $L_0^X(\Omega)$ such that (u_n) converges in probability to ξ. Since $u_{n+1}=\Phi u_n$ and Φ is continuous in probability, letting $n\to\infty$, we get $\Phi\xi=\xi$, i.e., ξ is a random fixed point of Φ.

Let η be another random fixed point of Φ. So, for any $t>0$, if $\mathbb{P}(\|\xi-\eta\|>t)>0$, then we have

$$\mathbb{P}(\|\xi-\eta\|>t)=\mathbb{P}\big(\big\|\Phi^n\xi-\Phi^n\eta\big\|>t\big)$$
$$\leq f_n\big(\mathbb{P}(\|\xi-\eta\|>t)\big)$$

for all n. Letting $n\to\infty$, we have

$$\mathbb{P}(\|\xi-\eta\|>t)<f\big(\mathbb{P}(\|\xi-\eta\|>t)\big)$$
$$<\mathbb{P}(\|\xi-\eta\|>t)$$

which yields a contradiction. So, we have $\mathbb{P}(\|\xi-\eta\|>t)=0$ for any $t>0$, i.e., $\xi=\eta$ a.s. Thus, Φ has a unique random fixed point. □

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