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g-Navier-Stokes Equations with Infinite Delays

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Abstract. We study the first initial boundary value problem for the two-dimensional non-autonomous g-Navier-Stokes equations containing infinite delay terms in an arbitrary (bounded or unbounded) domain satisfying the Poincaré inequality. The existence and uniqueness of a weak solution to the problem is proved by using the Galerkin method. Moreover, we also analyze the stationary problem and, under suitable additional conditions, we obtain global exponential decay of the solution of the evolutionary problem to the stationary solution.

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1. Introduction

Let Ω be a (bounded or unbounded) domain in $\mathbb{R}^2$ with boundary Γ. In this paper we study the existence and long-time behavior of solutions to the following two-dimensional non-autonomous g-Navier-Stokes equations with infinite delays:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f(t) + F(t, u_t) \text{ in } (\tau, T) \times \Omega, \\ \nabla \cdot (gu) &= 0 \text{ in } (\tau, T) \times \Omega, \\ u &= 0 \text{ on } (\tau, T) \times \Gamma, \\ u(\tau + s, x) &= \phi(s, x), \ s \in (-\infty, 0], x \in \Omega, \end{cases} \tag{1}$$

where $u = u(x,t) = (u_1, u_2)$ is the unknown velocity vector, $p = p(x,t)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient.

The g-Navier-Stokes equations is a variation of the standard Navier-Stokes equations. More precisely, when $g \equiv$ const we get the usual Navier-Stokes equations. The 2D g-Navier-Stokes equations arise in a natural way when we study the standard 3D problem in thin domains. We refer the reader to [14] for a derivation of the 2D g-Navier-Stokes equations from the 3D Navier-Stokes equations and a relationship between them. As mentioned in [10], good properties of the 2D g-Navier-Stokes equations can initiate the study of the Navier-Stokes equations on the thin three-dimensional domain $\Omega_g = \Omega \times (0, g)$. Therefore, in the last few years, the existence and asymptotic behavior of solutions to g-Navier-Stokes equations have been studied extensively (see e.g. [1, 2, 8, 9, 10, 14]).

However, there are situations in which the model is better described if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system (in a certain sense) by applying a force which takes into account not only the present state, but the complete history of the solutions. Therefore, in this paper we are interested in the case in which terms containing infinite delays appear. It is noticed that equations of Navier-Stokes type with delays in bounded domains has been studied in [3, 4, 5, 6] for the case of finite delays and very recently in [11, 12] for the case of infinite delays. One new feature in this paper is that we are able to prove the existence and global stability of solutions of 2D g-Navier-Stokes equations in an infinite delay case and domains that are not necessarily bounded but satisfy the Poincaré inequality. The obtained results, in particular, extend and improve some recent ones for Navier-Stokes equations with infinite delays in bounded domains [11] and for g-Navier-Stokes equations without delays [1].

It is known that there are numerous technical difficulties in dealing with partial differential equations with infinite delays in unbounded domains due to the unboundedness of the delay involved, and because the Sobolev embeddings are no longer compact. These introduce a major obstacle for proving the existence of solutions. To overcome these difficulties, in this paper we try to combine the techniques used for Navier-Stokes equations in unbounded domains (see e.g. [15, 6]) and the techniques used in [11] in dealing with the infinite delays.

Let X be a Banach space. Given a function $u : (-\infty, T) \to X$, for each $t < T$ we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t+s), s \in (-\infty, 0]$.

One possibility to deal with infinite delays, and which we will use here, is to consider, for any $\gamma > 0$, the space

$$C_\gamma(H_g) = \{\varphi \in C((-\infty, 0]; H_g) : \exists \lim_{s \to -\infty} e^{\gamma s}\varphi(s) \in H_g\},$$

which is a Banach space with the norm

$$\|\varphi\|_\gamma := \sup_{s \in (-\infty, 0]} e^{\gamma s}|\varphi(s)|.$$

Here the space H_g is defined in Section 2 below and $|\cdot|$ denotes the norm in H_g.

In order to study problem (1), we make the following assumptions:

(H1) The domain Ω can be an arbitrary (bounded or unbounded) domain in $\mathbb{R}^2$ without any regularity assumption on its boundary Γ, provided that the Poincaré inequality holds on Ω: There exists $\lambda_1 > 0$ such that

$$\int_\Omega \phi^2 g dx \leq \frac{1}{\lambda_1} \int_\Omega |\nabla\phi|^2 g dx \quad \forall \phi \in H_0^1(\Omega);$$

(H2) $g \in W^{1,\infty}(\Omega)$ such that

$$0 < m_0 \leq g(x) \leq M_0 \text{ for all } x = (x_1, x_2) \in \Omega, \text{ and } |\nabla g|_\infty < m_0 \lambda_1^{1/2};$$

(H3) $f \in L^2(\tau, T; V_g')$, where V_g' is the dual of the space V_g defined in Section 2;

(H4) $F(t, u_t) : (\tau, T) \times C_\gamma(H_g) \to L^2(\Omega, g)$ such that

(i) $\forall \xi \in C_\gamma(H_g)$, the mapping $(\tau, T) \ni t \mapsto F(t, \xi)$ is measurable,

(ii) $F(t, 0) = 0$ for all $t \in (\tau, T)$,

(iii) there exists a constant $L_F > 0$ such that $\forall t \in (\tau, T)$ and $\xi, \eta \in C_\gamma(H_g)$:

$$|F(t, \xi) - F(t, \eta)| \leq L_F \|\xi - \eta\|_\gamma.$$

Here the space $L^2(\Omega, g)$ is defined in Section 2 below.

We now give an example of the delay term $F(t, u_t)$. Let $F : (\tau, T) \times C_\gamma(H_g) \to L^2(\Omega, g)$ be defined as follows

$$F(t, \xi) = \int_{-\infty}^{0} G(t, s, \xi(s)) ds \quad \forall t \in (\tau, T), \xi \in C_\gamma(H_g),$$

where the function $G : (\tau, T) \times (-\infty, 0) \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following assumptions:

1. $G(t, s, 0) = 0$ for all $(t, s) \in (\tau, T) \times (-\infty, 0)$;
2. There exists a function $\kappa : (-\infty, 0) \to (0, \infty)$ such that

$$\|G(t, s, u) - G(t, s, v)\|_{\mathbb{R}^2} \leq \kappa(s)\|u - v\|_{\mathbb{R}^2}$$
$$\forall u, v \in \mathbb{R}^2, \forall (t, s) \in (\tau, T^*) \times (-\infty, 0),$$

and the function κ satisfies that $\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot} \in L^2(-\infty,0)$ for some $\varepsilon > 0$.

Then the function F satisfies (H4). Indeed, (H4-i) and (H4-ii) are obviously satisfied, for (H4-iii) we have

$$
\begin{aligned}
&|F(t,\xi) - F(t,\eta)|^2 \\
&= \int_\Omega \Big(\int_{-\infty}^0 \kappa(s)\|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2} ds \Big)^2 dx \\
&\le \int_\Omega \Big(\int_{-\infty}^0 \kappa^2(s)e^{-2(\gamma+\varepsilon)s} ds \Big)\Big(\int_{-\infty}^0 e^{2(\gamma+\varepsilon)s}\|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 ds \Big) dx \\
&= \|\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot}\|_{L^2(-\infty,0)}^2 \int_{-\infty}^0 \int_\Omega e^{2(\gamma+\varepsilon)s}\|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 dx ds \\
&\le \|\kappa(\cdot)e^{-(\gamma+\varepsilon)}\|_{L^2(-\infty,0)}^2 \Big[\sup_{s\in(-\infty,0]} e^{2\gamma s}\int_\Omega \|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^2}^2 dx \Big] \int_{-\infty}^0 e^{2\varepsilon s} ds \\
&= \|\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot}\|_{L^2(-\infty,0)}^2 \|\xi - \eta\|_\gamma^2 \frac{1}{2\varepsilon} \\
&= L_F^2 \|\xi - \eta\|_\gamma^2 .
\end{aligned}
$$

The rest of the paper is organized as follows. In the next section, we recall some auxiliary results on function spaces and inequalities for the nonlinear terms, which are related to the g-Navier-Stokes equations. In Section 3, we prove the existence of a weak solution to problem (1) by using the Galerkin method. The existence, uniqueness and global stability of a stationary solution are studied in the last section under some additional conditions.

2. Preliminary results

Let $L^2(\Omega, g) = (L^2(\Omega))^2$ and $H_0^1(\Omega, g) = (H_0^1(\Omega))^2$ be endowed, respectively, with the inner products

$$(u, v)_g = \int_\Omega u \cdot v g dx, \quad u, v \in L^2(\Omega, g),$$

and

$$((u,v))_g = \int_\Omega \sum_{j=1}^2 \nabla u_j \cdot \nabla v_j g dx, \quad u = (u_1, u_2), v = (v_1, v_2) \in H_0^1(\Omega, g),$$

and norms $|u|^2 = (u,u)_g$, $\|u\|^2 = ((u,u))_g$. Thanks to assumption (H2), the norms $|\cdot|$ and $\|\ \|$ are equivalent to the usual ones in $(L^2(\Omega))^2$ and in $(H_0^1(\Omega))^2$.

Let

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^2 : \nabla \cdot (gu) = 0\}.$$

Denote by H_g the closure of $\mathcal{V}$ in $L^2(\Omega, g)$, and by V_g the closure of $\mathcal{V}$ in $H_0^1(\Omega, g)$. It follows that $V_g \subset H_g \equiv H_g' \subset V_g'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V_g', and $\langle\, ,\cdot\rangle$ for duality pairing between V_g and V_g'.

We now define the trilinear form b by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j g dx,$$

whenever the integrals make sense. It is easy to check that if $u, v, w \in V_g$, then

$$b(u, v, w) = -b(u, w, v).$$

Hence

$$b(u, v, v) = 0 \text{ and } b(u, u, u - v) - b(v, v, u - v) = b(u - v, v, u - v) \ \forall u, v \in V_g.$$

Set $A : V_g \to V_g'$ by $\langle Au, v\rangle = ((u, v))_g$, $B : V_g \times V_g \to V_g'$ by $\langle B(u, v), w\rangle = b(u, v, w)$. Denote $D(A) = \{u \in V_g : Au \in H_g\}$, then $D(A) = H^2(\Omega, g) \cap V_g$ and $Au = -P_g\Delta u\ \forall u \in D(A)$, where P_g is the ortho-projector from $L^2(\Omega, g)$ onto H_g.

Using the Hölder inequality, the Ladyzhenskaya inequality (when $n = 2$):

$$|u|_{L^4} \leq c|u|^{1/2}|\nabla u|^{1/2} \ \forall u \in H_0^1(\Omega),$$

and the interpolation inequalities, as in [15] one can prove the following

Lemma 2.1. *If $n = 2$, then*

$$|b(u, v, w)| \leq \begin{cases} c_1|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2} & \forall u, v, w \in V_g, \\ c_2|u|^{1/2}\|u\|^{1/2}\|v\||Aw|^{1/2}|w|^{1/2} & \forall u \in V_g, v \in D(A), w \in H_g, \\ c_3|u|^{1/2}|Au|^{1/2}\|v\||w| & \forall u \in D(A), v \in V_g, w \in H_g, \\ c_4|u|\|v\||w|^{1/2}|Aw|^{1/2} & \forall u \in H_g, v \in V_g, w \in D(A), \end{cases} \tag{2}$$

where $c_i, i = 1, \dots, 4$, are appropriate constants.

Lemma 2.2. [2] *Let $u \in L^2(\tau, T; V_g)$, then the function Bu defined by*

$$(Bu(t), v)_g = b(u(t), u(t), v) \ \forall v \in V_g, \ a.e.\ t \in [\tau, T],$$

belongs to $L^2(\tau, T; V_g')$.

Lemma 2.3. [2] *Let $u \in L^2(\tau, T; V_g)$, then the function Cu defined by*

$$(Cu(t), v)_g = ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = b(\frac{\nabla g}{g}, u, v) \ \forall v \in V_g,$$

belongs to $L^2(\tau, T; H_g)$, *and hence also belongs to* $L^2(\tau, T; V_g')$. *Moreover,*

$$|Cu(t)| \leq \frac{|\nabla g|_\infty}{m_0} \cdot |u(t)| \quad \text{for a.e. } t \in (\tau, T),$$

and

$$\|Cu(t)\|_* \leq \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \cdot \|u(t)\| \quad \text{for a.e. } t \in (\tau, T)$$

Since

$$-\frac{1}{g}(\nabla \cdot g\nabla)u = -\Delta u - (\frac{\nabla g}{g} \cdot \nabla)u,$$

we have

$$(-\Delta u, v)_g = ((u, v))_g + ((\frac{\nabla g}{g} \cdot \nabla)u, v)_g = (Au, v)_g + (Cu, v)_g \quad \forall u, v \in V_g.$$

Denote by $\mathcal{V}(\mathcal{O})$ the same space as $\mathcal{V}$ but with an open set $\mathcal{O}$ instead of Ω, and analogously define $V_g(\mathcal{O})$ the closure of $\mathcal{V}(\mathcal{O})$ in $H_0^1(\mathcal{O}, g)$, $H_g(\mathcal{O})$ the closure of $\mathcal{V}(\mathcal{O})$ in $L^2(\mathcal{O}, g)$, and $D(A(\mathcal{O})) = H^2(\mathcal{O}, g) \cap V_g(\mathcal{O})$.

3. Existence and uniqueness of weak solutions

Definition 3.1. A weak solution on the interval (τ, T) of problem (1) is a function $u \in C((-\infty, T]; H_g) \cap L^2(\tau, T; V_g)$ with $u_\tau = \phi$, and such that for all $v \in V_g$,

$$\frac{d}{dt}(u(t), v)_g + \nu((u(t), v))_g + b(u(t), u(t), v) + \nu(Cu(t), v)_g = \langle f(t), v\rangle + (F(t, u_t), v)_g, \tag{3}$$

in the sense of $\mathcal{D}'(\tau, T)$.

It is noticed that if u is a weak solution of (1), then u satisfies the following energy equality

$$\begin{aligned} &|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr + 2\nu \int_s b(\frac{\nabla g}{g}, u(r), u(r))dr \\ &= |u(s)|^2 + 2\int_s^t \Big[\langle f(r), u(r)\rangle + (F(r, u_r), u(r))_g\Big] dr. \end{aligned}$$

Theorem 3.2. *Suppose that* $\phi \in C_\gamma(H_g)$ *is given and that* $2\gamma > \nu\lambda_1\gamma_0$, *where* $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} > 0$. *Then, there exists a unique weak solution* u *of problem* (1) *on the interval* (τ, T).

Proof. (i) *Uniqueness.* Let u, v be two weak solutions of problem (1) with the same initial condition and set $w = u - v$. Then, using the energy equality, we obtain

$$|w(t)|^2 + 2\nu \int_\tau^t \|w(s)\|^2 ds + 2\nu \int_\tau^t b(\frac{\nabla g}{g}, w(s), w(s))ds$$
$$= -2\int_\tau^t b(w(s), v(s), w(s))ds + 2\int_\tau^t (F(s, u_s) - F(s, v_s), w(s))_g ds.$$

By Lemmas 2.1 and 2.3, we have

$$\left|2\int_\tau^t b(w(s), v(s), w(s))ds\right| \le 2c_1 \int_\tau |w(s)|\|w(s)\|\|v(s)\|ds$$
$$\le \nu \int_\tau^t \|w(s)\|^2 ds + \frac{c_1^2}{\nu}\int_\tau \|v(s)\|^2 |w(s)|^2 ds$$

and

$$\left|2\nu\int_\tau^t b(\frac{\nabla g}{g}, w(s), w(s))ds\right| \le 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\int_\tau^t \|w(s)\||w(s)|ds$$
$$\le \nu\int_\tau^t \|w(s)\|^2 ds + \frac{\nu|\nabla g|_\infty^2}{m_0^2\lambda_1}\int_\tau |w(s)|^2 ds.$$

Because of (H4-iii), we have

$$\left|2\int_\tau^t (F(s, u_s) - F(s, v_s), w(s))ds\right| \le 2\int_\tau^t |F(s, u_s) - F(s, v_s)||w(s)|ds$$
$$\le 2L_F \int_\tau \|w_s\|_\gamma |w(s)|ds.$$

Since $w(s) = 0\ \forall s \le \tau$, we have

$$\|w_s\|_\gamma = \sup_{\theta\le 0} e^{\gamma\theta}|w(s+\theta)|$$
$$\le \sup_{\theta\in[\tau-s,0]} e^{\gamma\theta}|w(s+\theta)| \quad \text{for } \tau \le s \le T.$$

Therefore, one has

$$|w(t)|^2 \le \frac{c_1^2}{\nu}\int_\tau \|v(s)\|^2|w(s)|^2 ds + \left(2L_F + \frac{\nu|\nabla g|_\infty^2}{m_0^2\lambda_1}\right)\int_\tau^t \sup_{r\in[\tau,s]}|w(r)|^2 ds.$$

Hence we deduce that

$$\sup_{r\in[\tau,t]}|w(r)|^2 \le \int_\tau^\cdot \left(2L_F + \frac{\nu|\nabla g|_\infty^2}{m_0^2\lambda_1} + \frac{c_1^2}{\nu}\|v(s)\|^2\right)\sup_{r\in[\tau,s]}|w(r)|^2 ds,$$

whence the Gronwall inequality completes the proof of uniqueness.

(ii) *Existence.* We split the proof of the existence into several steps.

Step 1. A Galerkin scheme. Since V_g is separable and $\mathcal{V}$ is dense in V_g, there exists a sequence of linearly independent elements $\{v_1, v_2, ...\} \subset V$ which is total in V_g. Denote $V_m = \text{span}\{v_1, ..., v_m\}$ and consider the projector $P_m u = \sum_{j=1}^m (u, v_j)v_j$. Define also

$$u^m(t) = \sum_{j=1}^m \alpha_{m,j}(t)v_j,$$

where the coefficients $\alpha_{m,j}$ are required to satisfy the following system

$$\begin{aligned}&\frac{d}{dt}(u^m(t), v_j)_g + \nu\langle Au^m(t), v_j\rangle + \nu(Cu^m(t), v_j)_g + b(u^m(t), u^m(t), v_j)\\ &= \langle f(t), v_j\rangle + (F(t, u_t^m), v_j)_g \quad \forall j = 1, ..., m,\end{aligned} \tag{4}$$

and the initial condition $u^m(\tau + s) = P_m\phi(s)$ for $s \in (-\infty, 0]$.

The above system of ordinary functional differential equations with infinite delay in the unknown $(\alpha_{m,1}(t), ..., \alpha_{m,m}(t))$ fulfills the conditions for existence and uniqueness of local solutions (see [7, Theorem 1.1, p. 36]), so the approximate solutions u_m exist.

Step 2. A priori estimates. Multiplying (4) by $\alpha_{m,j}(t)$ and summing in j, we obtain

$$\begin{aligned}&\frac{d}{dt}(u^m(t), u^m(t))_g + \nu\langle Au^m(t), u^m(t)\rangle + \nu(Cu^m(t), u^m(t))_g\\ &\quad + b(u^m(t), u^m(t), u^m(t)) = \langle f(t), u^m(t)\rangle + (F(t, u_t^m), u^m(t))_g.\end{aligned} \tag{5}$$

Because $b(u^m(t), u^m(t), u^m(t)) = 0$ and $(Cu^m(t), u^m(t))_g = b(\frac{\nabla g}{g}, u^m(t), u^m(t))$, from (5) we have

$$\begin{aligned}&\frac{d}{dt}(u^m(t), u^m(t))_g + \nu\langle Au^m(t), u^m(t)\rangle + \nu b(\frac{\nabla g}{g}, u^m(t), u^m(t))\\ &\langle f(t), u^m(t)\rangle + (F(t, u_t^m), u^m(t))_g\end{aligned}$$

and therefore,

$$\begin{aligned}\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 &= 2\langle f(t), u^m(t)\rangle + 2(F(t, u_t^m), u^m(t))_g\\ &\quad - 2\nu b(\frac{\nabla g}{g}, u^m(t), u^m(t))\end{aligned} \tag{6}$$

Using the Cauchy inequality and Lemma 2.3, we get

$$\begin{aligned}\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 &\le 2\epsilon\nu\|u^m(t)\|^2 + \frac{\|f(t)\|_*^2}{2\epsilon\nu} + 2L_F\|u_t^m\|_\gamma^2\\ &\quad + 2\nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u^m(t)\|^2\end{aligned}$$

and hence

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu(\gamma_0 - \epsilon)\|u^m(t)\|^2 \le 2\Big(\frac{\|f(t)\|_*^2}{4\epsilon\nu} + L_F\|u_t^m\|_\gamma^2\Big), \tag{7}$$

where $\gamma_0 = 1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}} > 0$ and $\epsilon > 0$ is chosen such that $\gamma_0 - \epsilon > 0$. Noting that $\|u^m(t)\|^2 \ge \lambda_1|u^m(t)|^2$, we also have

$$\frac{d}{dt}|u^m(t)|^2 + \nu\lambda_1(\gamma_0 - \epsilon)|u^m(t)|^2 + \nu(\gamma_0 - \epsilon)\|u^m(t)\|^2 \le 2\Big(\frac{\|f(t)\|_*^2}{4\epsilon\nu} + L_F\|u_t^m\|_\gamma^2\Big).$$

Hence

$$|u^m(t)|^2 + \nu(\gamma_0 - \epsilon)\int_\tau^t e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-s)}\|u^m(s)\|^2 ds$$
$$\le e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-s)}|u^m(\tau)|^2 + 2\int_\tau^t e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-s)}\Big[\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big]ds. \tag{8}$$

Furthermore,

$$\|u_t^m\|_\gamma^2 \le \max\Big\{\sup_{\theta\in(-\infty,\tau-t]} e^{2\gamma\theta}|\phi(\theta+t-\tau)|^2, \sup_{\theta\in[\tau-t,0]}\Big[e^{2\gamma\theta-\nu\lambda_1(\gamma_0-\epsilon)(t-\tau+\theta)}|u(\tau)|^2$$
$$+ 2e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1(\gamma_0-\epsilon)(t+\theta-s)}\Big(\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big)ds\Big]\Big\}.$$

On one hand,

$$\sup_{\theta\in(-\infty,\tau-t]} e^{\gamma\theta}|\phi(\theta+t-\tau)| = \sup_{\theta\le 0} e^{\gamma(\theta-(t-\tau))}|\phi(\theta)| = e^{-\gamma(t-\tau)}\|\phi\|_\gamma.$$

On the other hand, as we are assuming that $2\gamma > \nu\lambda_1\gamma_0$,

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta-\nu\lambda_1(\gamma_0-\epsilon)(t-\tau+\theta)}|u(\tau)|^2 \le e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-\tau)}|u(\tau)|^2$$

and

$$\sup_{\theta\in[\tau-t,0]} e^{2\gamma\theta}\int_\tau^{t+\theta} e^{-\nu\lambda_1(\gamma_0-\epsilon)(t+\theta-s)}\Big(\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big)ds$$
$$\le \int_\tau^{t+\theta} e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-s)}\Big(\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big)ds.$$

Combining these inequalites we deduce that

$$\|u_t^m\|_\gamma^2 \le e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-\tau)}\|\phi\|_\gamma^2 + 2\int_\tau^t e^{-\nu\lambda_1(\gamma_0-\epsilon)(t-s)}\Big(\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big)ds.$$

By the Gronwall lemma we have

$$\|u_t^m\|_\gamma^2 \le e^{-[\nu\lambda_1(\gamma_0-\epsilon)-2L_F](t-\tau)}\|\phi\|_\gamma^2 + \frac{1}{2\epsilon\nu}\int_\tau^t e^{-[\nu\lambda_1(\gamma_0-\epsilon)-2L_F](t-s)}\|f(s)\|_*^2 ds.$$

Then we obtain the following estimates: for any $R > 0$ such that $\|\phi\|_\gamma \leq R$, there exists a constant C_1 depending on $\lambda_1, \nu, L_F, \epsilon, f, R, \tau$, such that

$$\|u_t^m\|_\gamma^2 \leq C_1 \quad \forall t \in [\tau, T], m \geq 1. \tag{9}$$

In particular, this implies that

$$\{u^m\} \text{ is bounded in } L^\infty(\tau, T; H_g). \tag{10}$$

Integrating (7) from τ to T, we have

$$\begin{aligned} |u^m(T)|^2 + 2\nu(\gamma_0 - \epsilon)\int_\tau^T \|u^m(s)\|^2 ds &\leq |u(\tau)|^2 + 2\int_\tau^T \Big[\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F\|u_s^m\|_\gamma^2\Big] ds \\ &\leq R^2 + 2\int_\tau^T \Big[\frac{\|f(s)\|_*^2}{4\epsilon\nu} + L_F C_1\Big] ds, \end{aligned}$$

thus, there exists a constant C_2 depending on R, C_1 such that

$$\|u^m\|_{L^2(\tau,T,V_g)}^2 \leq C_2 \quad \forall m \geq 1. \tag{11}$$

This implies that $\{u^m\}$ is bounded in $L^2(\tau, T, V_g)$.

Now, observe that (4) is equivalent to

$$\frac{du^m}{dt} = -\nu A u^m - \nu C u^m - P_m B(u^m, u^m) + P_m f(t) + P_m F(t, u_t^m). \tag{12}$$

Hence, we have

$$\{(u^m)'\} \text{ is bounded in } L^2(\tau, T, V_g') \tag{13}$$

So, there exist $u \in L^\infty(\tau, T; H_g) \cap L^2(\tau, T; V_g)$ with $u' \in L^2(\tau, T, V_g')$ and a subsequence of $\{u^m\}$, relabelled the same, such that

- $\{u^m\}$ converges weakly-star to u in $L^\infty(\tau, T, H_g)$,
- $\{u^m\}$ converges weakly to u in $L^2(\tau, T; V_g)$,
- $\{(u^m)'\}$ converges weakly to u' in $L^2(\tau, T; V_g')$.

If Ω is bounded, then the Aubin-Lions lemma in [13, Chapter 1] allows us to obtain a compactness result: a subsequence u^m converges to u in $L^2(\tau, T; H_g)$. If Ω is unbounded, we will have a similar result but not in a straightforward way, nor on the whole domain Ω. Actually, what holds in this case is the following: For any bounded open set $\mathcal{O} \subset \Omega$ there exists a subsequence (depending on $\mathcal{O}$ which we relabel) satisfying

$$u^m|_{\mathcal{O}} \to u|_{\mathcal{O}} \text{ in } L^2(\tau, T; (L^2(\mathcal{O}, g)). \tag{14}$$

For the sake of clarity, we postpone the proof to Lemma 3.4 below. Then we can pass to the limit in the term $b(u^m, u^m, \cdot)$ thanks to the following lemma whose proof is exactly the proof of Lemma 3 2 in [15, Chapter III].

Lemma 3.3. *If u_m converges to u in $L^2(\tau, T; V_g(\mathcal{O}))$ weakly and in $L^2(\tau, T; H_g(\mathcal{O}))$ strongly, where $\mathcal{O}$ is an open bounded set, then for any vector function w with components belonging to $C^1(\overline{\mathcal{O}})$, we have*

$$\int_\tau^T b(u_m(t), u_m(t), w(t))dt \to \int_\tau^T b(u(t), u(t), w(t))dt.$$

However, the estimates obtained above are not enough to pass to the limit in the term $F(t, u_t^m)$.

Step 3. Convergence in $C_\gamma(H_g(\mathcal{O}))$ and existence of a weak solution.

We will prove that

$$u_t^m \to u_t \text{ in } C_\gamma(H_g(\mathcal{O})) \quad \forall t \in (-\infty, T].$$

It is not difficult to check that this holds if we prove the following

$$P_m\phi \to \phi \text{ in } C_\gamma(H_g(\mathcal{O})), \tag{15}$$

$$u^m \to u \text{ in } C([\tau, T], H_g(\mathcal{O})) \tag{16}$$

Step 3.1. Approximation in $C_\gamma(H_g(\mathcal{O}))$ of the initial datum.

We now check the convergence claimed in (15). Indeed, if not, there would exist $\epsilon > 0$ and a subsequence, that we relabel the same, such that

$$e^{\gamma\theta_m}|P_m\phi(\theta_m) - \phi(\theta_m)| > \epsilon. \tag{17}$$

One can assume that $\theta_m \to -\infty$, otherwise if $\theta_m \to \theta$, then $P_m\phi(\theta_m) \to \phi(\theta)$, since $|P_m\phi(\theta_m) - \phi(\theta| \le |P_m\phi(\theta_m) - P_m\phi(\theta)| + |P_m\phi(\theta) - \phi(\theta)| \to 0$ as $m \to +\infty$ But with $\theta_m \to -\infty$ as $m \to +\infty$, if we denote $x = \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta)$, we obtain that

$$\begin{aligned} e^{\gamma\theta_m}|P_m\phi(\theta_m) - \phi(\theta_m)| &= |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - e^{\gamma\theta_m}\phi(\theta_m)| \\ \le |P_m(e^{\gamma\theta_m}\phi(\theta_m)) - P_m x| &+ |P_m x - x| + |x - e^{\gamma\theta_m}\phi(\theta_m)| \to 0. \end{aligned}$$

This is a contradiction with (17), so (15) holds.

Step 3.2. Convergence of u^m to u in $C([\tau, T]; H_g(\mathcal{O}))$.

From the strong convergence of $\{u^m\}$ to u in $L^2(\tau, T; H_g(\mathcal{O}))$, we deduce that

$$u^m(t) \to u(t) \text{ in } H_g(\mathcal{O}) \text{ a.e. } t \in (\tau, T).$$

Since

$$u^m(t) - u^m(s) = \int_s^t (u^m)'(r)dr \text{ in } V_g'(\mathcal{O}) \quad \forall s, t \in [\tau, T],$$

from (13) we have that $\{u^m\}$ is equi-continuous on $[\tau, T]$ with values in $V_g'(\mathcal{O})$. By the compactness of the embeding $H_g(\mathcal{O}) \subset V_g'(\mathcal{O})$, from (10) and the equi-continuity in $V_g'(\mathcal{O})$, using the Arzela-Ascoli theorem we have

$$u^m \to u \text{ in } C([\tau, T]; V_g'(\mathcal{O})). \tag{18}$$

Again from (10) we obtain that for any sequence $\{t_m\} \subset [\tau, T]$ with $t_m \to t$,

$$u^m(t_m) \to u(t) \text{ weakly in } H_g(\mathcal{O}), \tag{19}$$

where we have used (18) in order to identify which is the weak limit.

Now, we are ready to prove (16) by a contradiction argument. If it would not be so, then taking into account that $u \in C([\tau, T]; H_g(\mathcal{O}))$, there would exist $\epsilon > 0$, a value $t_0 \in [\tau, T]$ and subsequences (relabelled the same) $\{u^m\}$ and $\{t_m\} \subset [\tau, T]$ with $\lim_{m\to+\infty} t_m = t_0$ such that

$$|u^m(t_m) - u(t_0)| \geq \epsilon \quad \forall\, m. \tag{20}$$

To prove that this is absurd, we will use an energy method. Observe that the following energy inequality holds for all u^m:

$$\begin{aligned} &\frac{1}{2}|u^m(t)|^2 + \nu\Big(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\Big) \int_s^t \|u^m(r)\|^2 dr \\ &\leq \int_s^t \langle f(r), u^m(r)\rangle dr + \frac{1}{2}|u^m(s)|^2 + C_3(t - s) \quad \forall\, s, t \in [\tau, T], \end{aligned} \tag{21}$$

where $C_3 = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |F(r, u_r^m)|^2 dr \leq D(t - s) \quad \forall\, \tau \leq s < t \leq T$$

On the other hand, from (10), (H4-ii), (H4-iii), there exists $\xi_F \in L^2(\tau, T; L^2(\mathcal{O}, g))$ such that $\{F(t, u^m)\}$ converges weakly to ξ_F in $L^2(\tau, T; L^2(\mathcal{O}, g))$. Thus, we can pass to the limit in equation (12) and deduce that u is a solution of

$$\frac{d}{dt}(u(t), v)_g + \nu((u(t), v))_g + \nu(Cu(t), v)_g + b(u(t), u(t), v) = \langle f(t), v\rangle + (\xi_F(t), v)_g. \tag{22}$$

Therefore, u satisfies the energy equality

$$\begin{aligned} &|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 dr + 2\nu \int_s^t (Cu(r), u(r))_g dr \\ &= |u(s)|^2 + 2\int_s^t (\langle f(r), u(r)\rangle + (\xi_F(r), u(r))_g) dr \quad \forall s, t \in [\tau, T], \end{aligned}$$

and for the weak limit ξ_F we have the estimate

$$\int_s^t |\xi_F|^2 dr \leq \liminf_{m\to+\infty} \int_s^t |F(r, u_r^m)|^2 dr \leq D(t - s) \quad \forall \tau \leq s \leq t \leq T.$$

So, we have that u also satisfies inequality (21) with the same constant C_3. Now, consider two functions $J_m, J : [\tau, T] \to \mathbb{R}$ defined by

$$J_m(t) = \frac{1}{2}|u^m(t)|^2 - \int_\tau^t \langle f(r), u^m(r)\rangle dr - C_3 t,$$

$$J(t) = \frac{1}{2}|u(t)|^2 - \int_\tau^t \langle f(r), u(r)\rangle dr - C_3 t.$$

It is clear that J_m and J are non-increasing and continuous functions. Moreover, by the convergence of u^m to u a.e. in time with value in $H_g(\mathcal{O})$, and weakly in $L^2(\tau, T; H_g(\mathcal{O}))$, it holds that

$$J_m(t) \to J(t) \text{ a.e. } t \in [\tau, T]. \tag{23}$$

Now we will prove that

$$u^m(t_m) \to u(t_0) \text{ in } H_g(\mathcal{O}), \tag{24}$$

which contradicts (20). First, recall from (19) that

$$u^m(t_m) \rightharpoonup u(t_0) \text{ weakly in } H_g(\mathcal{O}), \tag{25}$$

so we have

$$|u(t_0)| \leq \liminf_{m\to+\infty} |u^m(t_m)|.$$

Therefore, if we show that

$$\limsup_{m\to+\infty} |u^m(t_m)| \leq |u(t_0)|, \tag{26}$$

we will obtain that $\lim_{m\to+\infty} |u^m(t_m)| = |u(t_0)|$, which jointly with (25) imply (24).

Now, observe that the case $t_0 = \tau$ follows directly from (21) with $s = \tau$ and the definition of $u^m(\tau) = P_m\phi(0)$. So, we may assume that $t_0 > \tau$. This is important, since we will approach this value t_0 from the left by a sequence $\{t'_k\}$, i.e. $\lim_{k\to+\infty} t'_k \nearrow t_0$. Since $u(.)$ is continuous at t_0, there is k_ϵ such that

$$|J(t'_k) - J(t_0)| < \frac{\epsilon}{2} \quad \forall\, k \geq k_\epsilon.$$

On the other hand, taking $m \geq m(k_\epsilon)$ such that $t_m > t'_{k_\epsilon}$, as J_m is non-increasing and for all t'_k the convergence (24) holds, one has

$$J_m(t_m) - J(t_0) \leq |J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| + |J(t'_{k_\epsilon}) - J(t_0)|,$$

and obviously, taking $m \leq m'(k_\epsilon)$, it is possible to obtain $|J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| < \frac{\epsilon}{2}$. It can also be deduced from Step 2 that

$$\int_\tau^{t_m} \langle f(r), u^m(r)\rangle dr \to \int_\tau^{t_0} \langle f(r), u(r)\rangle dr,$$

so we conclude that (26) holds. Thus, (24) and finally (16) are also true, as we wanted to check. Hence, we have

$$F(\cdot, u^m) \to F(\cdot, u) \text{ in } L^2(\tau, T; L^2(\mathcal{O}, g)). \tag{27}$$

In what follows we will show that the convergence results above enable us to conclude that u is a solution of problem (1). Let ψ be a continuously differentiable function on $[0, T]$. Multiplying (1) by $\psi(t)$, we have

$$\int_\tau^T \left(\frac{du^m(t)}{dt}, v_j\psi(t)\right)_g dt + \nu\int_\tau^T \langle Au^m(t), v_j\psi(t)\rangle dt$$
$$+ \nu\int_\tau^T (Cu^m(t), v_j\psi(t))_g dt + \int_\tau^T b(u^m(t), u^m(t), v_j\psi(t))dt$$
$$\int_\tau^T \langle f(t), v_j\psi(t)\rangle dt + \int_\tau^T (F(t, u_t^m), v_j\psi(t))_g dt.$$

Taking a diagonal subsequence, denote again as u^m, that satisfies (14) and (27) for a sequence of regular bounded open sets $\mathcal{O}_j \subset \Omega$ that contain all supports of functions v_j of the basis. Passing to the limit, we have

$$\int_\tau^T \left(\frac{du(t)}{dt}, v_j\psi(t)\right)_g dt + \nu\int_\tau^T \langle Au(t), v_j\psi(t)\rangle dt$$
$$+ \nu\int_\tau^T (Cu(t), v_j\psi(t))_g dt + \int_\tau^T b(u(t), u(t), v_j\psi(t))dt$$
$$\int_\tau^T \langle f(t), v_j\psi(t)\rangle dt + \int_\tau^T (F(t, u_t), v_j\psi(t))_g dt$$

holds for all v_j in the basis and any continuously differentiable function ψ on $[0, T]$. Thus, we see that u satisfies (3) in the distribution sense. ∎

At the end of this section, we prove the following lemma, which has been used in the proof of Theorem 3.2.

Lemma 3.4. *Under the assumptions of Theorem 3.2, the sequence u^m given in (4) is precompact in the following sense: suppose a bounded open set $\mathcal{O} \subset \Omega$ is given, then there exists a subsequence depending on $\mathcal{O}$, which we relabel, such that*

$$u^m|_{\mathcal{O}} \to u|_{\mathcal{O}} \text{ in } L^2(\tau, T; L^2(\mathcal{O}, g)),$$

where u is the limit given in (14).

To prove Lemma 3.4, we will use the following

Lemma 3.5. [11, Theorem 2.2] *Let Θ be a bounded open set of $\mathbb{R}$ and $X \subset E$ be Banach spaces with compact injection. Consider $1 \leq r < q \leq \infty$. Suppose $F \subset L^r(\Theta; E)$ satisfies*

(i) $\forall \omega \subset\subset \Theta$, $\sup_{f\in F} \|\tau_h f - f\|_{L^r(\omega,E)} \to 0$ *when* $h \to 0$, *where* $\tau_h f$ *is the translation*

$$(\tau_h f)(x) = f(x+h).$$

(ii) F *is bounded in* $L^q(\Theta; E) \cap L^1(\Theta; X)$

Then F *is precompact in* $L^r(\Theta; E)$.

Proof of Lemma 3.4. Fix $\chi \in C^1(\mathbb{R}_+)$ with $\chi(s) = 1$ for $s \in [0,1]$ and $\chi(s) = 0$ for $s \geq 4$. Consider $\mathcal{O}$ as in the statement, let $R > 0$ be such that $\mathcal{O} \in B(0,R)$ and denote $\mathcal{O}' = \Omega \cap B(0,2R)$, and $u^{m,R}(x) = u^m(x)\chi(|x|^2/R^2)$. Again the compactness holds for $X = H_0^1(\mathcal{O}', g) \subset E = L^2(\mathcal{O}', g)$ with compact injection, and we conserve the original u^m on $\Omega \cap B(0,R)$.

For the sake of clarity, we continue the proof directly with u^m instead of $u^{m,R}$. Since condition (ii) in Lemma 3.5 is obviously satisfied by (10) and (11), we concentrate on (i). Actually, we will prove that for the whole domain Ω the following property holds:

$$\sup_{m\in\mathbb{N}} \|\tau_h u^m - u^m\|_{L^2(0,T-h;L^2(\Omega,g))} \to 0 \text{ when } h \to 0.$$

Consider $h > 0$ arbitrarily small. From (4) we deduce for $(t, t+h) \subset (\tau, T)$ that

$$\begin{aligned}
&\int_\Omega (u^m(t+h) - u(t))w_j g dx + \nu \int_t^{t+h} \int_\Omega \nabla u^m(s) \cdot \nabla w_j g dx ds \\
&+ \nu \int_t^{t+h} b(\frac{\nabla g}{g}, u^m(s), w_j) ds + \int_t^{t+h} b(u^m(s), u^m(s), w_j) ds \\
&= \int_t^{t+h} \int_\Omega f(s) w_j g dx ds + \int_t^{t+h} F(s, u_s^m) w_j g dx ds.
\end{aligned}$$

Multiplying by $\gamma_{mj}(t+h) - \gamma_{mj}(t)$ and summing in j we obtain

$$\begin{aligned}
\int_\Omega |u^m(t+h) - u(t)|^2 g dx &= -\nu \int_t^{t+h} \int_\Omega \nabla u^m(s)(\nabla u^m(t+h) - \nabla u^m(t)) g dx ds \\
-\nu \int_t^{t+h} b\Big(\frac{\nabla g}{g}, u^m(s), u^m(t+h) - u^m(t)\Big) ds &- \int_t^{t+h} b(u^m(s), u^m(s), u^m(t+h) - u^m(t)) ds \\
+ \int_t^{t+h} \int_\Omega f(s)\cdot(u^m(t+h) - u^m(t)) g dx ds &+ \int_t^{t+h} \int_\Omega F(s, u_s^m).(u^m(t+h) - u^m(t)) g ds.
\end{aligned}$$

The right-hand side may be bounded by

$$\begin{aligned}
&\nu |\nabla u^m(t+h) - \nabla u^m(t)| \int_t^{t+h} |\nabla u^m(s)| ds \\
&+ \nu \int_t^{t+h} \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}} \|u^m(s)\| |u^m(t+h) - u^m(t)| ds
\end{aligned}$$

$$+ \int_t^{t+h} c|u^m(s)|\|u^m(s)\|\|u^m(t+h) - u^m(t)\|ds$$

$$+ \int_t^{t+h} \|f(s)\|_*\|u^m(t+h) - u^m(t)\|ds + \int_t^{t+h} |F(s, u_s^m)||u^m(t+h) - u^m(t)|ds.$$

Thus, using (H2) and (10), we have proved that

$$\int_\Omega |u^m(t+h) - u^m(t)|^2 g dx \leq \|u^m(t+h) - u^m(t)\| \int_t^{t+h} G_m(s)ds,$$

where the function $G_m : \mathbb{R} \to \mathbb{R}$ is defined as:

$$G_m(s) = \nu\|u^m(s)\| + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1}\|u^m(s)\| + cK_1\|u^m(s)\| + \|f(s)\|_* + \lambda_1^{-1/2}|F(s, u^m(s))|,$$

with K_1 being a constant independent of m such that $|u^m(s)| \leq K_1$.

To finish the proof, we will estimate

$$\begin{aligned}\|\tau_h u^m - u^m\|^2_{L^2(\tau,T-h;L^2(\Omega,g))} &= \int_\tau^{T-h}\int_\Omega |\tau_h u^m - u^m|^2 g dx dt\\ &\leq \int_\tau^{T-h} \|u^m(t+h) - u^m(t)\| \int_t^{t+h} G_m(s)dsdt.\end{aligned}$$

For the right-hand side, the Fubini theorem yields, using the function

$$\overline{s} = \begin{cases} 0 & \text{if } s \leq 0, \\ s & \text{if } 0 < s \leq T-h, \\ T-h & \text{if } s > T-h, \end{cases}$$

to

$$\begin{aligned}&\int_\tau^{T-h} \|u^m(t+h) - u^m(t)\| \int_t^{t+h} G_m(s)dsdt\\ &\leq \int_\tau^T G_m(s) \int_{\overline{s-h}}^{\overline{s}} \|u^m(t+h) - u^m(t)\|dtds \leq 2(hK_2)^{1/2}\int_\tau^T G_m(s)ds,\end{aligned}$$

where K_2 is a constant independent of m such that $\int_\tau^T \|u^m(s)\|^2 ds \leq K_2$, and we have used the Young inequality and the facts that

$$0 \leq \overline{s} - \overline{s-h} \leq h \text{ for } \int_{\overline{s-h}}^{\overline{s}} \|u^m(t+h) - u^m(t)\|\, dt,$$

and

$$\int_{\overline{s-h}}^{\overline{s}} \|u^m(t+h) - u^m(t)\|\, dt \leq \left(\int_{\overline{s-h}}^{\overline{s}} dt\right)^{1/2}\left(\int_{\overline{s-h}}^{\overline{s}} \|u^m(t+h) - u^m(t)\|\, dt\right)^{1/2}$$

$$\leq 2h^{1/2}\Big(\int_{\tau}^{T-h}\int_{\Omega}|\nabla u^m|^2 g dx dt\Big)^{1/2} \leq 2h^{1/2}K_2^{1/2}.$$

To conclude, we observe that $\int_{\tau}^{T} G_m(s)ds$ is bounded. Indeed, one has

$$\int_{\tau}^{T} G_m(s)ds = \int_{\tau}^{T}\Big[\Big(\nu+\nu\frac{|\nabla g|_\infty}{m_0\lambda_1}+cK_1\Big)\|u^m(s)\|+\|f(s)\|_*+\lambda_1^{-1/2}|F(s,u_s^m)|\Big]ds$$

$$\leq \Big(\nu+\nu\frac{|\nabla g|_\infty}{m_0\lambda_1}+cK_1\Big)\sqrt{T-\tau}\Big(\int_{\tau}^{T}\|u^m(s)\|^2ds\Big)^{1/2}$$

$$+\sqrt{T-\tau}\Big(\int_{\tau}^{T}\|f(s)\|_*^2ds\Big)^{1/2}+\sqrt{T-\tau}\lambda_1^{-1/2}\Big(\int_{\tau}^{T}|F(s,u_s^m)|ds\Big)^{1/2}$$

and assumptions (H3)-(H4) give the bound for the two last terms. ■

4. Existence and stability of stationary solutions

In this section, we will study the existence and stability of a stationary solution to problem (1) under some additional conditions.

The restrictions we must impose to give sense to a stationary solution are that $f \in V_g'$ and F are now autonomous, i.e. without dependence on time, and we must clarify how F acts over a fixed element of H_g. This is done with a slight abuse of notation in the following sense: We consider $F(w)$ as $F(w')$, where $w' \in C_\gamma(H_g)$ is the element that has the only value w for time $t \leq 0$. Of course, as an immediate consequence of the assumptions for F, it follows that

$$|F(x_1)-F(x_2)| \leq L_F|x_1-x_2| \quad \forall x_1,x_2 \in H_g.$$

So, consider the following equation

$$\frac{du}{dt}+\nu Au+\nu Cu+B(u,u)=f+F(u_t) \quad \forall t\in(\tau,T). \tag{28}$$

A stationary solution to problem (28) is an element $u^* \in V_g$ such that

$$\nu((u^*,v))_g+\nu(Cu^*,v)_g+b(u^*,u^*,v)=\langle f,v\rangle+(F(u^*),v)_g \quad \forall v\in V_g. \tag{29}$$

Theorem 4.1. *Under the above assumptions and notations, if*

$$\nu\Big(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\Big) > \frac{L_F}{\lambda_1},$$

then

(a) *Problem (28) admits at least one stationary solution u^*. Moreover, any such stationary solution satisfies the estimate*

$$\left[\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}\right]\|u^*\| \leq \|f\|_*. \tag{30}$$

(b) *If the following condition holds*

$$\left[\nu\left(1 - \frac{|\nabla g|_\infty}{m_0 \lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}\right]^2 > \frac{c_1}{\lambda_1^{1/2}}\|f\|_*, \tag{31}$$

where c_1 is the constant in Lemma 2.1, then the stationary solution of (28) is unique.

Proof. (i) *Existence.* The estimate (30) can be obtained taking into account that in particular any stationary solution u^*, if it exists, should verify

$$\nu\langle Au^*, u^*\rangle + \nu(Cu^*, u^*)_g = \langle f, u^*\rangle + (F(u^*), u^*)_g$$

and therefore

$$\nu\|u^*\|^2 \leq \|f\|_* \|u^*\| + \frac{L_F}{\lambda_1}\|u^*\|^2 + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u^*\|^2.$$

For the existence, since V_g is separable there exists a sequence of linearly independent elements $v_1, v_2, \ldots$ which is total in V_g. For each $m \geq 1$, let us denote $V_m = \mathrm{span}\{v_1, \ldots, v_m\}$ and we would like to define an approximate solution u^m of (28) by

$$u^m = \sum_{i=1}^m \gamma_{mi} v_i,$$

$$\nu((u^m, v_i)) + \nu b\left(\frac{\nabla g}{g}, u^m, v_i\right) + b(u^m, u^m, v_i) = \langle f, v_i\rangle + (F(u^m), v_i)_g, \; i = 1, \ldots, m. \tag{32}$$

To prove the existence of u^m, we define operators $R_m : V_m \to V_m$ by

$$((R_m u, v)) = \nu\langle Au, v\rangle + \nu(Cu, v)_g + b(u, u, v) - \langle f, v\rangle - (F(u), v)_g \;\; \forall u, v \in V_m.$$

For all $u \in V_m$,

$$\begin{aligned}((R_m u, u)) &= \nu\langle Au, u\rangle + \nu(Cu, u)_g - \langle f, u\rangle - (F(u), u)_g \\ &\geq \nu\|u\|^2 - \|f\|_*\|u\| - \frac{L_F}{\lambda_1}\|u\|^2 - \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u\|^2 \\ &= \left(\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}\right)\|u\|^2 - \|f\|_*\|u\|.\end{aligned}$$

Thus, if we take

$$\beta = \frac{\|f\|_*}{\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}},$$

we obtain $((R_m u, u)) \geq 0$ for all $u \in V_m$ such that $\|u\| = \beta$. Consequently, by a corollary of the Brouwer fixed point theorem (see [15, Chapter 2, Lemma 1.4]), for each $m \geq 1$ there exists $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \leq \beta$. Replacing v_i by u^m in (32) and taking into account that $b(u^m, u^m, u^m) = 0$, we get

$$\begin{aligned}\nu\|u^m(t)\|^2 &= (f, u^m) + (F(u^m), u^m)_g - \nu b(\frac{\nabla g}{g}, u^m, u^m)\\ &\leq \|f\|_* \|u^m\| + \frac{L_F}{\lambda_1}\|u^m\|^2 + \nu\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u^m\|^2.\end{aligned}$$

Hence

$$\left[\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}) - \frac{L_F}{\lambda_1}\right]\|u^m\| \leq \|f\|_*. \tag{33}$$

We extract from $\{u^m\}$ a sequence $\{u^{m'}\}$, which converges weakly in V_g to some limit u. If Ω is bounded, then the injection of V_g into H_g is compact. Thus, this convergence holds also in the norm of H_g

$$u^{m'} \to u \text{ weakly in } V_g \text{ and strongly in } H_g,$$

up to a subsequence. Passing to the limit in (32) with the sequence m', we find that u is a weak solution of (28). In the case that Ω is unbounded, the injection of V_g into H_g is no longer compact. However, this difficulty can be overcome by using arguments as in [15, p. 168-171].

(ii) *Uniqueness.* Suppose that u^* and v^* are two stationary solutions of (28). Then

$$\nu(Au^* - Av^*, v) + b(u^*, u^*, v) - b(v^*, v^*, v) + \nu(Cu^* - Cv^*, v)_g = (F(u^*) - F(v^*), v)_g$$

for all $v \in V_g$. Taking $v = u^* - v^*$, we have

$$\nu(Au^* - Av^*, u^* - v^*) = b(v, v^*, v) - \nu(Cu^* - Cv^*, u^* - v^*)_g + (F(u^*) - F(v^*), u^* - v^*)_g.$$

Hence

$$\nu\|u^* - v^*\|^2 \leq c_1\lambda_1^{-1/2}\|u^* - v^*\|^2\|v^*\| + \frac{L_F}{\lambda_1}\|u^* - v^*\|^2 + \frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|u^* - v^*\|^2$$

and therefore

$$\left[\nu(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}) - \frac{L_F}{\lambda_1}\right]\|u^* - v^*\|^2 \leq c_1\lambda_1^{-1/2}\|u^* - v^*\|^2\|v^*\|. \tag{34}$$

From (30) and (34) we have

$$\left[\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}\right]^2 \|u^* - v^*\|^2 \le c_1\lambda_1^{-1/2}\|f\|_*\|u^* - v^*\|^2, \tag{35}$$

and the uniqueness follows from (31) and (35). ■

Theorem 4.2. *Assume that the assumptions in Theorem 3.2 with f and F independent of time and (31) hold. Then there exists a value $\lambda \in (0, 2\gamma)$ such that for the solution $u(t)$ of (1) with $\tau = 0$ and $\phi \in C_\gamma(H_g)$, the following estimates hold for all $t \ge 0$*

$$|u(t) - u^*|^2 \le e^{-\lambda t}\left(|\phi(0) - u^*|^2 + \frac{L_F}{2\gamma - \lambda}\|\phi - u^*\|_\gamma^2\right), \tag{36}$$

$$\|u_t - u^*\|_\gamma \le \max\left\{e^{-2\gamma t}\|\phi - u^*\|_\gamma^2, e^{-\lambda t}\left(|\phi(0) - u^*|^2 + \frac{L_F}{2\gamma - \lambda}\|\phi - u^*\|_\gamma^2\right)\right\}, \tag{37}$$

where u^* is the unique stationary solution of (28).

Proof. Denote $w(t) = u(t) - u^*$, one has

$$\frac{d}{dt}(w(t), v)_g + \nu((w(t), v))_g + \nu(Cu(t), v)_g - \nu(Cu^*, v)_g + b(u(t), u(t), v)$$
$$-b(u^*, u^*, v) = (F(u_t) - F(u^*), v)_g \quad \forall t > 0, v \in V_g.$$

From the energy equality, (H4-iii), Lemmas 2.1 and 2.3, and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, we obtain

$$\frac{d}{dt}(e^{\lambda t}|w(t)|^2) = e^{\lambda t}\Big[\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + 2\nu(Cu^* - Cu(t), w(t))_g$$
$$+2(b(u^*, u^*, w(t)) - b(u(t), u(t), w(t))) + 2(F(u_t) - F(u^*), w(t))_g\Big]$$
$$\le e^{\lambda t}\Big[\lambda|w(t)|^2 - 2\nu\|w(t)\|^2 + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\|w(t)\|^2$$
$$+ \frac{2c_1}{\lambda_1^{1/2}}\|w(t)\|^2\|u^*\| + 2L_F\|u_t\|_\gamma|w(t)|\Big].$$

Hence, using the Cauchy inequality with $\delta > 0$ to be fixed later on and (30), we have

$$\frac{d}{dt}(e^{\lambda t}|w(t)|^2) \le e^{\lambda t}\frac{L_F}{\delta}\|w_t\|_\gamma^2$$
$$+ e^{\lambda t}\left[\lambda\lambda_1^{-1} - 2\nu + \frac{\delta L_F}{\lambda_1} + \frac{2c_1\|f\|_*}{\lambda_1^{1/2}\left(\nu\left(1 - \frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right) - \frac{L_F}{\lambda_1}\right)} + \frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right]\|w(t)\|^2.$$

Therefore, integrating from 0 to t, we have

$$e^{\lambda t}|w(t)|^2 \le |w(0)|^2 + \frac{L_F}{\delta}\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds$$

$$+\left[\lambda\lambda_1^{-1}-2\nu+\frac{\delta L_F}{\lambda_1}+\frac{2c_1\|f\|_*}{\lambda_1^{1/2}\left(\nu(1-\frac{\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}})-\frac{L_F}{\lambda_1}\right)}+\frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}\right]\int_0^t e^{\lambda s}\|w(s)\|^2ds. \tag{38}$$

In order to control the term $\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds$, we proceed as follows

$$\int_0^t e^{\lambda s}\sup_{\theta\le 0} e^{2\gamma\theta}|w(s+\theta)|^2ds$$

$$\int_0^t e^{\lambda s}\max\{\sup_{\theta\le -s} e^{2\gamma\theta}|w(s+\theta)|^2,\ \sup_{\theta\in[-s,0]} e^{2\gamma\theta}|w(s+\theta)|^2\}ds$$

$$\int_0^t \max\{e^{-(2\gamma-\lambda)s}\|\phi-u^*\|_\gamma^2,\ \sup_{\theta\in[-s,0]} e^{(2\gamma-\lambda)\theta}e^{\lambda(s+\theta)}|w(s+\theta)|^2\}ds.$$

So, if $\lambda<2\gamma$, using the above equality in (38), we obtain

$$e^{\lambda t}|w(t)|^2\le|w(0)|^2+\frac{L_F}{\delta}\|\phi-u^*\|_\gamma^2\int_0^t e^{(\lambda-2\gamma)s}ds+\left[\lambda\lambda_1^{-1}-2\nu+\frac{\delta L_F}{\lambda_1}\right.$$

$$\left.+\frac{2c_1\|f\|_*}{\lambda_1^{1/2}\left(\nu(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})-\frac{L_F}{\lambda_1}\right)}+\frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}+\frac{L_F}{\lambda_1\delta}\right]\int_0^t\max_{r\in[0,s]}e^{\lambda r}\|w(r)\|^2ds.$$

Observe that the choice of $\delta=1$ makes that $\delta\lambda_1^{-1}L_F+L_F(\lambda_1\delta)^{-1}$ is minimal and the coefficient of the last integral becomes

$$\lambda\lambda_1^{-1}-2\nu+\frac{2L_F}{\lambda_1}+\frac{2c_1\|f\|_*}{\lambda_1^{1/2}\left[\nu(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})-\frac{L_F}{\lambda_1}\right]}+\frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}. \tag{39}$$

Using (31), we have

$$-2\nu+\frac{2L_F}{\lambda_1}+\frac{2c_1\|f\|_*}{\lambda_1^{1/2}\left[\nu(1-\frac{|\nabla g|_\infty}{m_0\lambda_1^{1/2}})-\frac{L_F}{\lambda_1}\right]}+\frac{2\nu|\nabla g|_\infty}{m_0\lambda_1^{1/2}}<0.$$

Thus, we can choose $\lambda\in(0,2\gamma)$ such that (39) is negative. So, we can deduce that

$$e^{\lambda t}|w(t)|^2\le|w(0)|^2+\frac{L_F}{2\gamma-\lambda}(1-e^{(\lambda-2\gamma)t})\|\phi-u^*\|_\gamma^2,$$

whence (36) follows.

Finally, (37) can be deduced as follows

$$\|w_t\|_\gamma^2=\sup_{\theta\le 0}e^{2\gamma\theta}|w(t+\theta)|^2$$

$$=\max\left\{\sup_{\theta\in(-\infty,-t]}e^{2\gamma\theta}|w(t+\theta)|^2,\ \sup_{\theta\in[-t,0])}e^{2\gamma\theta}|w(t+\theta)|^2\right\}$$

$$= \max\left\{e^{-2\gamma t}\|\phi - u^*\|_\gamma^2, \sup_{\theta\in[-t,0]} e^{2\gamma\theta}|w(t+\theta)|^2\right\}$$

and the second term can be estimated using (36) and the fact that $e^{(2\gamma-\lambda)\theta} \le 1$ when $\theta \le 0$. ∎

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